AN EXPLICIT EXAMPLE OF TOMTER'S ALGEBRAIC ANOSOV FLOWS OF THE MIXED TYPE

We give an example of Tomter's algebraic Anosov flow of mixed type over a geodesic flow of a genus 2 surface, and try to understand the orbits and foliations in the fibres.

1 Arithmetic Fuchsian groups

We know that $\mathcal{A} := (\frac{26,-5}{\mathbb{Q}})$ is a quaternion division algebra that gives rise to an arithmetic Fuchsian group of a genus 2 surface [Mac08]. We compute the action on the torus $\mathbb{R}^4$. Then the action of all the integer points passes to an action on $\mathbb{T}^4$.

More generally, for a quaternion algebra $\mathcal{B} = (\frac{a,b}{\mathbb{F}})$ where $\mathbb{F}$ is a field extension of degree n over $\mathbb{Q}$, $\mathcal{B}$ acts on $\mathbb{R}^{4n}$.

We can write $x \in \mathcal{A}$ as $x = a + bi + cj + dk$ where $i^2 = 26$ and $j^2 = -5$, $a, b, c, d \in \mathbb{Q}$. We also have $ij = k = -ji, jk = -kj = jij = -ij^2 = 5i, ik = -ki = iij = 26j$, $k^2 = ijij = -jiij = -26j^2 = 130$.

There is an embedding of $x = a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$ into $M_{22}(\mathbb{R})$ by

$$1 \mapsto \begin{bmatrix} 1 & \\ & 1 \end{bmatrix},\ i \mapsto \begin{bmatrix} \sqrt{26} & \\ & -\sqrt{26} \end{bmatrix},\ j \mapsto \begin{bmatrix} & 1 \\ -5 & \end{bmatrix},\ k \mapsto \begin{bmatrix} & \sqrt{26} \\ 5\sqrt{26} & \end{bmatrix}.$$

Then $x \mapsto \begin{bmatrix} a + b\sqrt{26} & c + d\sqrt{26} \\ -5c + d(5\sqrt{26}) & a - b\sqrt{26} \end{bmatrix}$. Note that $|x|^2 = x\bar{x} = (a + bi + cj + dk)(a - bi - cj - dk) = a^2 - 26b^2 + 5c^2 - 130d^2 = \det\left(\begin{bmatrix} a + b\sqrt{26} & c + d\sqrt{26} \\ -5c + d(5\sqrt{26}) & a - b\sqrt{26} \end{bmatrix}\right)$.

We can also embed $\mathcal{A}$ into $M_{44}(\mathbb{Q})$ by $x \mapsto (M_x : y \mapsto xy)$ for $x, y \in \mathcal{A}$. Now M_x is 4-by-4 matrix with rational coefficients which we can write down explicitly:

$$\begin{bmatrix} a & 26b & -5c & 130d \\ b & a & 5d & 5c \\ c & -26d & a & 26b \\ d & -c & b & a \end{bmatrix}.$$

The image of $\mathcal{A}$ forms a linear subspace of $M_{44}(\mathbb{Q})$.

On the other hand, we consider $\begin{bmatrix} a + b\sqrt{26} & c + d\sqrt{26} \\ -5c + d(5\sqrt{26}) & a - b\sqrt{26} \end{bmatrix}$ acting on $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ (a change of basis of $1, i, j, k \in \mathcal{A}$ in $M_{22}(\mathbb{R})$) as acting on $\mathbb{R}^2 \oplus \mathbb{R}^2 = \mathbb{R}^4$, so the $\det(M_x) = \det\left(\begin{bmatrix} a + b\sqrt{26} & c + d\sqrt{26} \\ -5c + d(5\sqrt{26}) & a - b\sqrt{26} \end{bmatrix}\right)^2$.

Now denote $\mathbb{G}(F) := \{M_x \in M_{44}(F) \cap W : \det(M_x) = 1\}$, where F can be $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$. We check that $\mathbb{G}(\mathbb{R}) \cong SL(2, \mathbb{R})$: the isomorphism is given by $(x \mapsto M_x)$. Denote $\Gamma := \mathbb{G}(\mathbb{Z})/\{\pm I_2\}$. Γ is a Fuchsian group and it is cocompact in $PSL(2, \mathbb{R})$ by Mahler compactness criterion [Kat92, Theorem 5.4.1]. The action of $\Gamma = (\mathbb{Z})/\{\pm I_2\}$ on the torus is well-defined by taking $\pm x \mapsto \pm M_{\pm x}$.

2 Tomter's example

We compute the eigenvalues and eigenspaces of such matrices M_x. Let also $a^2 - 26b^2 + 5c^2 - 130d^2 = 1$.

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For any $\gamma \in \Gamma$, there exists $g \in PSL(2,\mathbb{R})$ such that $g\gamma g^{-1}$ is a translation along the axis (i, i) of the upper half plane model $\mathbb{H}^2$, which corresponds to $\begin{bmatrix} a + b\sqrt{26} & \\ & a - b\sqrt{26} \end{bmatrix}$ (in this case $c = d = 0$). The eigenvalues of $M_{g\gamma g^{-1}}$ are $\lambda_1 = a + \sqrt{26}b$, $\lambda_2 = a - \sqrt{26}b$, and the eigenvectors are $\begin{bmatrix} \sqrt{26} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \sqrt{26} \\ 1 \end{bmatrix}, \begin{bmatrix} -\sqrt{26} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\sqrt{26} \\ 1 \end{bmatrix}$. We have $\lambda_1\lambda_2 = 1$, and $M_{g\gamma g^{-1}}$ is hyperbolic. Therefore γ acts on the torus also by a hyperbolic matrix.

For any $(\gamma, n) \in \Gamma \ltimes \mathbb{Z}^4$, we have $(g, x)(\gamma, n) = (g\gamma, \gamma^{-1}.x + n)$. Denote $M := PSL(2,\mathbb{R}) \ltimes \mathbb{R}^4/(\Gamma \ltimes \mathbb{Z}^4)$. M is a torus bundle over the unit tangent bundle of a surface of genus 2.

Let $a_t = \begin{bmatrix} e^{-t/2} & \\ & e^{t/2} \end{bmatrix}$ denote the geodesic flow on $T^1\mathbb{H}^2 \cong PSL(2,\mathbb{R})$ and for any $g \in PSL(2,\mathbb{R})$ it acts by the right action $R_{a_t}(g) = ga_t^{-1}$. For $(g, x) \in PSL(2,\mathbb{R}) \ltimes \mathbb{R}^4$, let $R_{a_t}(g, x) = (ga_t^{-1}, x)$.

The flow $\Phi^t : M \to M$ defined by $\Phi^t((g, x)(\Gamma \ltimes \mathbb{Z}^4)) = (ga_t^{-1}, x)(\Gamma \ltimes \mathbb{Z}^4)$ is Anosov.

3 Orbits of Φ^t

References

[Kat92] Svetlana Katok. *Fuchsian groups*. Chicago: The University of Chicago Press, 1992.

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