Foliated Manifolds

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[1][2]The purpose of these eight lectures is to expose, without trying to be complete, the main results (actually very few) of the theory of foliated manifolds developed by C. Ehresmann and G.Reeb. Without aiming at the greater generality, we will try our best to clear up, by examples and proofs, the fundamental ideas of, for example, the holonomy which is due to C. Ehresmann and is the foundation of the entire theory.

The 3 basic references are

G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Act. Sc. et Ind., Hermann, Paris, 1952.

C. Ehresmann and Shih W.S., *Sur les espaces feuilletés: théorème de stabilité.* C. R. Acad. Sc. Paris, 243, 1956, p. 344-46.

A. Haefliger, *Structure feuilletées et cohomologie à valeur dans un faisceau de groupoïdes*, Comm. Math. Helv. 32, 1958, 248-329.

1 DEFINITIONS AND EXAMPLES

Let us say a function from an open set of the space of real numbers to another is of class r, where $r = 0, 1, 2, \dots, n, \dots, \infty$ or ω, if it is continuous when $r = 0$, if it is r times continuously differentiable when r is an integer > 0 or ∞, and if it is real analytic when r is the symbol ω.

A local homeomorphism[3] of $\mathbb{R}^n$ (i.e., a homeomorphism from an open set of $\mathbb{R}^n$ of dimension n to another open set of $\mathbb{R}^n$) is said to be of class r if itself is of class r and as well as its inverse.

A manifold of dimension n and class r is a topological space V, unless mentioned explicitly otherwise, endowed with a set of charts that are homeomorphisms from open sets of $\mathbb{R}^n$ to open sets of V, and the change of charts are also homeomorphisms of class r of $\mathbb{R}^n$

The notions of functions of class r or local homeomorphisms of class r extend naturally to the case of a manifold of class r.

A function f from a topological space X to a topological space Y is locally a homeomorphism if each point of X has an open neighborhood mapped homeomorphically under f to an open set in Y.

1.1 DEFINITION OF A FOLIATION IN TERMS OF CHARTS

Identify the real space $\mathbb{R}^n$ of dimension n with the product $\mathbb{R}^p \times \mathbb{R}^{n-p}$; denote by $x = (x_1, \dots, x_p)$ the first p coordinates of $\mathbb{R}^n$ and by $y = (y_1, \dots, y_{n-p})$ the last $n - p$ coordinates.

The simplest example of a foliated structure of codimension p on $\mathbb{R}^n$ is the one where the leaves are the $(n - p)$-planes parallel to the linear subspace defined by $x = 0$. A local homeomorphism h of class r of this structure $\mathcal{F}_0$ is a local homeomorphism of $\mathbb{R}^n$ that locally preserves the leaves: on an open

[1]This manuscript reproduces the series lectures given by the author during the C.I.M.E. course on *Differential Topology* held in Urbino, July 2-12, 1962. (Translator's note: This sentence was Google Translated from Italian.)

[2]Footnote by Danyu Zhang (zhang.8939@osu.edu; DZ later for short) who translated this: The bibliography of the original version is [Hae62]. I bounced between keeping the wording and structure of sentences of French, and making things into the language that I am more familiar with in English. The language still looks weird from time to time nonetheless. Sometimes if I was not sure if there was a typo or it was just that I did not understand, I kept the original and put a footnote. The numbering of sections, theorems etc. might be different from the original. I should say that I did not read carefully towards the end. Corrections and comments are all welcome.

[3]DZ: I kept "homeomorphism" throughout the note even if it might be better to call it a diffeomorphism, since sometimes when it says "of class r" the case of $r = 0$ is also included. Though at other times it is clear $r > 0$, I am too lazy to distinguish.

neighborhood of each point (x,y) where h is defined, the homeomorphism $h(x,y)=(x',y')$ is given by the equations of the form:

$$\begin{cases} x' = h_1(x) \\ y' = h_2(x,y). \end{cases} \tag{1}$$

Note that h_1 is a local homeomorphism of $\mathbb{R}^p$ of class r.

On a manifold of dimension n and class r, a foliated structure (or a foliation) $\mathcal{F}$ of class r and codimension p, or of dimension $n-p$, is defined by a maximal set (complete atlas) of charts $\{h_i\}$ that consists of homeomorphisms of class r from open sets U_i in $\mathbb{R}^n$ to open sets on V and which has the two properties:

(i) the images $h_i(U_i)$ of the charts h_i form an open cover of V,
(ii) each change of charts $h_j^{-1}h_i$ is a local homeomorphism of $\mathbb{R}^n$ of class r that is locally in the form of (1).

We also say that a foliation $\mathcal{F}$ is topological, differentiable or analytic if $r=0, 0<r\leq\infty$ or $r=\omega$ respectively.

Restricting to the form of the change of charts, we can define the structures even more precisely. For example the foliated structure is said to be *oriented* (or *transversly oriented*) if the change of charts is of the form (1), with an additional condition that the local homeomorphism h_1 of $\mathbb{R}^p$ preserves the orientation. It is said to be *transversly analytic* if h_1 is analytic.

We also have the notion of complex analytic foliations if we replace in the previous definition $\mathbb{R}^n$ by the complex space $\mathbb{C}^n$, and the change of charts are supposed to be complex analytic.

It is also clear that we can more generally replace in previous definition $\mathbb{R}^n=\mathbb{R}^p\times\mathbb{R}^{n-p}$ by the product $B\times F$ of any two topological spaces, and the change of charts are always in the form of (1).

Any foliated structure of class r on V obviously induces such a structure on any open set of V.

1.2 DISTINGUISHED FUNCTIONS

Let $\mathcal{F}$ be a foliated structure of codimension p on V, defined by a complete atlas that contains the charts h_i's and let π be the natural projection of $\mathbb{R}^n=\mathbb{R}^p\times\mathbb{R}^{n-p}$ onto the first factor $\mathbb{R}^p$. The continuous functions from the open sets of V to $\mathbb{R}^p$ that are locally of the form πh_i^{-1}[4] are called the *distinguished functions* of $\mathcal{F}$.

The dinstinguished functions form a set of functions $\{f_i\}$ of class r (and of rank p if $r>0$) on the open sets V_i of V to $\mathbb{R}^p$ which have the following properties:

a) the V_i's form an open cover of V
b) a function f on an open set U of V to $\mathbb{R}^p$ is distinguished if and only if, for each other distingushed function $f_i : V_i \to \mathbb{R}^p$ and each point $z\in U\cap V_i$, there exists a local homeomorphism h of class r of $\mathbb{R}^p$ such that

$$f = hf_i \qquad \text{in a neighborhood of } z. \tag{1'}$$

In the case of an oriented foliated structure, the local homeomorphism h of (1') is forced to preserve the orientation of $\mathbb{R}^p$.

Note that the distinguished functions completely characterize the structure $\mathcal{F}$ and we will use them constantly in the future.

Let f be a function from a manifold V of class r to another manifold V' of class r that is locally a homeomorphism of class r. Let $\mathcal{F}'$ be a foliation of class r on V'. The functions from the open sets of V to $\mathbb{R}^p$ obtained by composing f with the distinguished functions of $\mathcal{F}'$ are the distinguished functions of a foliation $\mathcal{F}$ of V called the *inverse image by f of $\mathcal{F}'$*.

[4]DZ: Now $h_i : V_i \to \mathbb{R}^p$, $V_i \subset V$.

Remark 1.1. The consideration of the distinguished functions leads to the following natural generalization.

Let B be a topological space with a pseudogroup Γ of transformations. A foliated Γ-structure on a topological space X is defined by a family of continuous functions on open sets V_i of V to B that satisfies the previous two conditions, and the local homeomorphism h in (1') must be an element of Γ.

The advantage of this definition is that it emphasizes the structure transverse to the leaves (that is the structure of B on which Γ acts) which is, according to our experience, often more important than that of the leaves.

1.3 THE LEAVES

Let T_0 be a topology of $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$ that is the product of the discrete topology of $\mathbb{R}^p$ and the natural topology of $\mathbb{R}^{n-p}$. The connected component of $\mathbb{R}^n$ under T_0 are precisely the leaves of $\mathcal{F}_0$, which are the planes x = constant, equipped with their natural topology. The local automorphisms h of $\mathcal{F}_0$ are also the homeomorphisms for the topology T_0 according to (1). Thus there exists a unique topology T on V such that each chart h_i is a homeomorphism from U_i to $h(U_i)$ for the topology induced by T_0 and T respectively.

By definition *the leaves of a structure $\mathcal{F}$ are the connected components of V relative to the topology T (called the topology of leaves).* The leaves of $\mathcal{F}$ are therefore the submanifolds of V of dimension $n-p$ that are of class r if $\mathcal{F}$ is of class r.

The topology of leaves T can also be defined by the distinguished functions: a basis of T is formed by the inverse images of points of $\mathbb{R}^p$ by the various distinguished functions.

The quotient space V by the equivalent relation ϱ, of which the classes are the leaves of $\mathcal{F}$, is called the *leaf space of* $\mathcal{F}$. Note that ϱ is an *open* equivalence relation[5] (the leaves that meet an open set of V form an open cover of V); in fact, it is generated by the local open equivalence relations ϱ_i of which the classes are the leaves of the foliated structures induced by $\mathcal{F}$ on the codomains of the charts h_i's. It follows that the closure of a subset of V which is a union of leaves is itself a union of leaves of $\mathcal{F}$.

We will often use the notion of the *transverse submanifold to a foliation.* Let $\mathcal{F}$ be a foliation of class r and codimension p on V. A submanifold W of class r of V (or a manifold W where there is a function j of class r into V) is said to be transverse to $\mathcal{F}$ if the restriction to W (or the composition with f) of each distinguished function of $\mathcal{F}$ is locally a homeomorphism of class r to $\mathbb{R}^p$. Thus for $r > 0$, at each point $z \in V$, the tangent space of W at z is complement to the tangent space of the leaf passing through z.

A leaf F is said to be *proper* if the induced topology on F from the topology of V is the same as the induced topology on F from the topology of leaves T.

From the above, a leaf is proper if and only if there exists a small transverse submanifold to the foliation intersectiong F at a single point.

A leaf F is *closed* if it is a closed subspace of V (with the usual topology). A proper leaf is closed if and only if each compact set in a transverse submanifold intersects F at a finite number of points, and conversely:

Proposition 1.2. *Each closed leaf of a foliation defined on a manifold V with a countable basis is also proper ([Hae56]).*

An example shows that on a manifold with an uncountable basis, a leaf can fill the entire manifold.

The following lemma was essentially proved in Chevalley [Che46] (see as well [Hae56], p. 4).

Lemma 1.3. *If V has a countable basis, each leaf of a foliation on V also has a countable basis.*

[5] DZ: Meaning the quotient map is open.

The proposition itself then remarks that if a closed leaf F is not proper, it would intersect a transverse submanifold in a closed perfect set. But a perfect set is not countable, which contradicts to the fact that F has a countable basis.

For more relevant properties of the general topology of foliations, see Reeb [WR52] Chapter A and [Ree56].

1.4 ORIENTATION OF A FOLIATION

Let $\mathcal{F}$ be a foliation of codimension p defined on V. The distinguished functions of $\mathcal{F}$ defined on a neighborhood of a point $z \in V$ are divided into two classes: two distinguished functions f and g are in the same class if the local homeomorphism h of $\mathbb{R}^p$ such that $g = hf$ to the neighborhood of z, is of degree 1 at $f(z)$. Each of the classes is called a *germ of orientation of $\mathcal{F}$ at the point z*.

We give the set V^* of the germs of orientation of $\mathcal{F}$ a topology by deciding that a subset U^* of V^* is an element of the basis of the topology if U^* is a set of germs of orientation defined by a distinguished function from different points of its domain. The projection of V^* onto V associates to each germ of orientation at point z, the point z itself, and makes V^* a two sheeted cover of V. More precisely, we have

Proposition 1.4. *The space V^* of germs of orientation of a foliation $\mathcal{F}$ on V is a two sheeted cover of V. The foliation $\mathcal{F}^*$ on V^*, the inverse image of $\mathcal{F}$ of the covering projection, is oriented. The foliation $\mathcal{F}$ is orientable if and only if the cover is trivial.*

Thus any foliation of a simply connected manifold is orientable.

EXAMPLES

1.5 SIMPLE FOLIATIONS

A foliation $\mathcal{F}$ of class r on a manifold V is said to be simple if there exists a function f from V to a manifold W (separated[6] or not) of class r and dimension p that satisfies the following condition: for each homeomorphism g of class r from an open set of $\mathbb{R}^p$ to an open set of W, the function $g^{-1}f$ is a distinguished function of $\mathcal{F}$. The leaves of $\mathcal{F}$ are then the connected components of the inverse images of the points of W under f. The leaf space of $\mathcal{F}$ (cfr. 1.2) is a manifold, in general not separated, of class r, with a projection φ onto W that is locally a homeomorphism of class r. Conversely, if the leaf space is a manifold of dimension p and of class r, then the foliation is simple. We can take for f the natural function from V to the leaf space of $\mathcal{F}$. A foliation is simple if and only if, for each point $z \in V$, there exists a distinguished function f defined in a neighborhood U of z that maps the non-empty intersection of any leaf with U to a single point of $\mathbb{R}^p$.

For example, by the implicit function theorem, a function f of class r $(r > 0)$ and rank p from V to a manifold of class r and dimension p defines on V a simple foliation.

Each foliated structure of class 0 and codimension 1 of a plane $\mathbb{R}^2$ is simple (cf. [HR57]).

On $\mathbb{R}^n$, every foliation of class 0 and codimension 1 all leaves of which are closed is also simple (cf. 3.4 and [Hae56]).

A fibration of class r on a manifold V is also an example of a simple foliation; the leaves are connected components of fibres. The induced structure on each open set of a simple foliation is simple.

1.6 COMPLETELY INTEGRABLE PLANE FIELDS

Let V be a manifold of class $r + 1 \geq 2$ and dimension n. Let Π be a q-plane field of class r on V. Alternatively, at each point $x \in V$ we are given a linear subspace $\Pi(x)$ of dimension q of the tangent vector space of V at x, and this subspace is a function of x of class r. The field Π is said to be completely

[6] DZ: Hausdorff.

integrable if, for each x of V, there exists a submanifold of class r of V of dimension q, such that at each point y of W, the tangent space of W at y is $\Pi(y)$.

If the field Π is locally given by q linearly independent vector fields $X_1, \ldots, X_q$ (i.e., at each point x where it is defined, the q vector fields X_i's generate $\Pi(x)$), and the condition of complete integrability is equivalent to the fact that the bracket $[X_i, X_j]$ of any two vector fields is a linear combination of the X_k's (cf. [Che46]). Dually, if Π is given locally by the annihilation of $n-q$ forms $\omega_1, \ldots, \omega_{n-q}$ of degree 1, the field Π is completely integrable if and only if the exterior differentials $d\omega_i$'s belong to the ring generated by the forms ω_j's.

If $\mathcal{F}$ is a foliated structure on V of class $r+1>1$ and dimension q, the tangent planes to the leaves of $\mathcal{F}$ form a completely integrable field of q-planes of class r.

Conversely, a completely integrable q-plane field of class r on V defines a foliated structure $\mathcal{F}$ of class r whose leaves are the maximal integral manifolds of this field. A distinguished function of $\mathcal{F}$ is given by $n-q$ locally independent first integrals.

Let us point out two fundamental problems, but about which nothing is known.

Existence problem. Let V be a manifold with a q-plane field Π of class r. Under what conditions does there exist on V a completely integrable q-plane field of class r that is homotopic to Π[7]?

Approximation problem. Let V be a manifold of class ∞ with a completely integrable q-plane field Π of class r. Under what conditions does there exist on V a completely integrable q-plane field of class $r' > r$ that approximates Π?

1.7 CLASSES MODULO A SUBGROUP

Let G be a Lie group and H a connected analytic subgroup of G. The left cosets of G modulo H are the leaves of an analytic foliated structure on G (cf. [Che46]). All the leaves are isomorphic to each other, because one can be obtained from a left translation of another. They are proper (cf 1.2) if and only if H is a closed subgroup. The simplest example of a subgroup that is not closed is obtained by taking for G the 2 dimensional torus (quotient of the plane $\mathbb{R}^2$ by the subgroup of points with integral coordinates) and for H the 1 parameter subgroup determined by a line passing through $(0,0)$ with irrational slope.

1.8 BUNDLE WITH DISCRETE STRUCTURAL GROUP

Let X and B be manifolds of class r, where X is connected and B is of dimension p. The manifold X is identified with the quotient of the universal cover $\tilde{X}$ by a group Π of homeomorphisms of B of class r, which is moreover isomorphic to the fundamental group of B[8]. Let Φ be a representation of Π into a group of homeomorphisms of B of class r. Let V be the quotient of the product $\tilde{X} \times B$ by the equivalence relation that identifies the couples (x', y') and (x, y) if there exists an element g of Π such that $x' = g(x)$ and $y' = \Phi(g)y$. The manifold $\tilde{X} \times B$, with the natural projection to V, is a covering of V.

The projection of $\tilde{X} \times B$ to $\tilde{X}$ gives a projection π of V onto X by passing to the quotient; with this projection, V is a fibre space with the base X, fibre B and discrete structural group Π (cf. [Ste51]) or with a integrable connection (cf. [Ehr52]). Conversely, each fibred structure of class r with discrete structure group can be obtained in this manner.

On the other hand, V has a foliated structure of class r and of codimension p whose leaves are transverse to the fibres. In fact, the projection of $\tilde{X} \times B$ onto B defines a simple foliated structure of class r on $\tilde{X} \times B$. The group Π acting on $\tilde{X} \times B$ by the transformations of the form $(x, y) \to (g(x), \Phi(g)y)$, where $g \in \Pi$, is a group of automorphisms of this foliation. By passing to the quotient, we obtain on V a foliated structure $\mathcal{F}_\Phi$ whose leaves are the images $F(y)$, by the natural projection of $\tilde{X} \times B$ to V, of

[7]DZ: Homotopic as sections of the tangent bundle.

[8]DZ: X.

the submanifolds $\tilde{X} \times y$, where $y \in B$. The manifold V, with the topology of leaves and the projection π to X, is a cover of X; in particular each leaf is a cover of X.

Here is a specific example of a foliation of codimension one obtained in this way on a manifold of dimension 3. Let X be a Klein bottle, quotient of the plane $\mathbb{R}^2$ by the group action Π generated by the two transformations

$$g_1 \begin{cases} x_1' = x_1 + 1 \\ x_2' = x_2 \end{cases} \quad \text{and} \quad g_2 \begin{cases} x_1' = -x_1 \\ x_2' = x_2 + 1 \end{cases}$$

This two generators are related by a single relation $g_1 g_2^{-1} g_1 g_2 =$ identity. Let Φ be a representation of Π into the group of homeomorphisms of the circle $\mathbb{S}^1$ with which g_1 is a non-periodic rotation and g_2 an axial symmetry (reversing the orientation). The previous construction gives a foliation on a orientable manifold of dimension 3; the leaves are either annuli or Möbius bands. Each leaf is everywhere dense.

Generally speaking, the study of foliations $\mathcal{F}_\Phi$ obtained in the previous way essentially reduces to the study of the representation Φ from the fundamental group of X into the homeomorphisms of B of class r.

Thus the leaf $F(y)$ corresponds to a orbit of the point y of B depending on the group $\Phi(\Pi)$; the leaf $F(y)$ is everywhere dense in V if and only if the orbit of y is everywhere dense in B; the leaf $F(y)$ is proper (cf. 1.3) if the topology of B induces the discrete topology on the orbit of y. The leaf space of $\mathcal{F}_\Phi$ is the quotient of B by the action of $\Phi(\Pi)$. When B is compact, the stability properties of $\mathcal{F}_\Phi$ and Φ coincide, etc.

Let us point out that G. Reeb made detailed study in the case where X is a torus, B is a segment $[0,1]$ and where $r \geq 2$ (cf. [Ree61]).

1.9 FOLIATION OF $\mathbb{S}^3$ OF CODIMENSION 1

Here is the simplest and most classic example that is due to Reeb. Let $\mathbb{D}^2$ be a closed disc of vectors x in the plane whose norm $|x|$ is ≤ 1. In the cylindar $\mathbb{D}^2 \times \mathbb{R}$, consider the foliation whose leaves are the boundary $\mathbb{S}^1 \times \mathbb{R}$ of the cylindar and whose other leaves are the surfaces defined by $(x, \frac{|x|^2}{1-|x|^2} + a)$, where a is a real parameter and $|x| < 1$.

A solid torus $\mathbb{T}$ is the quotient of the product $\mathbb{D}^2 \times \mathbb{R}$ by the equivalence relation that identifies (x,t) and $(x, t+n)$, where n is any integer; there is a one-to-one correspondence between foliations of $\mathbb{T}$ and foliations of $\mathbb{D}^2 \times \mathbb{R}$ invariant under the translation $(x,t) \mapsto (x, t+1)$. The previous foliation therefore defines a foliation in the solid torus whose boundary is a leaf.

Taking two foliated solid tori as above and gluing them along their boundaries by a homeomorphism that takes the meridians of one to the parallels of the other, we obtain a foliation on $\mathbb{S}^3$. All the leaves are proper and only one leaf is compact, namely the common boundary of the two tori.

Replacing the function $|x|^2/(1-|x|^2)$ by an appropriate function, we can obtain a foliation of class ∞; we will see later that there is no foliation on $\mathbb{S}^3$ that is analytic.

Choosing another foliation in the solid torus, the boundary is always a leaf, we can infintely vary the example of Reeb. We can obtain as many compact leaves as we want (they are always torus by a general theorem by Ehresmann (cf. [Ehr51])); in the known example, the non-compact leaves are homeomorphic to the planes with certain number of points taken; we can also obtain dense leaves in an open set.

Kneser has a conjecture that every foliation of $\mathbb{S}^3$ of codimension 1 admits a compact leaf. Among the open problems, let us mention the following: what are the knots in $\mathbb{S}^3$ whose boundary of a tubular neighborhood may be a leaf of a foliation? We can see that this is possible for torus knots. Is there a foliation of $\mathbb{S}^3$ with non-compact leaves that are not homeomorphic to punctured planes?

One may wonder also what the compact manifolds of dimension 3 that can be foliated by surfaces. It is always possible to construct a foliation of codimeison 1 on a 3-dimensional manifold that is a fibre space by the circles in the sense of Seifert (with singular fibres).

Let us remark, that apart from the case of $\mathbb{S}^3$, we do not know any foliation of spheres, besides the case where we know the existence of fibrations.

1.10 VORTEX OF A FOLIATION – (cf. Reeb, [WR52] p.114-5).

Let $\mathbb{D}^r$ be the closed disc of vectors z in $\mathbb{R}^r$ with the norm $|z| \leq 1$. In the product $\mathbb{S}^1 \times \mathbb{D}^{p-1} \times \mathbb{D}^{n-p}$ of the form of triples (θ, x, y), let us perturb the foliation of codimension p whose leaves are the $(n-p)$ discs $\{\theta\} \times \{x\} \times \mathbb{D}^{n-p}$ and leave the boundary fixed.

Embedding the tube $\mathbb{S}^1 \times \mathbb{D}^{p-1} \times \mathbb{D}^{n-p}$ into a foliated manifold in the manner that $\mathbb{S}^1 \times \mathbb{D}^{p-1} \times \{y\}$ is transverse to leaves and $(\theta, x, \mathbb{D}^{n-p})$ is contained in the leaves for each θ, x, y, we can perturb the foliation inside the tube without changing it outside.

Let $\alpha(u,v)$ be a function of class ∞ of two variables u and v, defined on the square $0 \leq |u| \leq 1$ and $0 \leq |v| \leq 1$, pair of u and v, equal to 0 in the neighborhood of the boundary and infinity at points $u = 0$ and $|v| = 1/2$.

In the product $\mathbb{R} \times \mathbb{D}^{p-1} \times \mathbb{D}^{n-p}$, formed by the triples (t, x, y), consider the foliation of class ∞ whose leaves are the submanifolds defined by

$$x = x_0, \qquad t = \alpha(|x|, |y|) + t_0, \qquad \text{where } x_0 \text{ and } t_0 \text{ are constants}$$

and by $x = 0$, $|y| = 1/2$.

The foliation is invariant by the translation $(t, x, y) \mapsto (t+1, x, y)$, so we get by passing to the quotient a foliation on $\mathbb{S}^1 \times \mathbb{D}^{p-1} \times \mathbb{D}^{n-p}$.

The leaves passing through the points where $x \neq 0$ are the $(n-p)$ discs. Those who pass through the points where $x = 0$ and $|y| \neq 1/2$ are homeomorphic to $\mathbb{R}^+ \times \mathbb{S}^{n-p-1}$ if $|y| > 1/2$ or to $\mathbb{R}^{n-p}$ if $|y| < 1/2$. There is a leaf homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{n-p-1}$, which is defined by $x = 0$ and $|y| = 1/2$. $\mathbb{R}^+$ denotes the half-line $[0, \infty]$[9].[10]

2 THE NOTION OF HOLONOMY

2.1 INTRODUCTION

Let $\mathcal{F}$ be a foliation of class r and codimension p on a manifold V. Let F be a leaf of $\mathcal{F}$ and $\pi_1(F, z)$ the fundamental group of F based at $z \in F$. The holonomy of F is a representation $\Phi : \pi_1(F, z) \to G$, where G is the group of germs of local homeomorphisms of class r of $\mathbb{R}^p$, at the origin 0, which leave 0 fixed (cf. 2.2). The representation is defined up to inner automorphisms of G. The image of Φ is the holonomy group of F (defined up to conjugacy).

This fundamental notion was introduced by C. Ehresmann (it was implicitly in Reeb's thesis); it gives almost all the information you want to know about the neighborhood of a leaf. For example the stability theorem says that if the holonomy group of a compact leaf is finite, then there exists an open neighborhood that is a union of compact leaves. In the differentiable or analytic case, we showed [Hae58] that the holonomy of a leaf itself completely charactrizes its foliated neighborhood (cf. 2.7).

The notion of holonomy is well known in the case of a dynamical system given on a manifold by a system of ordinary differential equations. Suppose F is a closed trajectory (a circle) and let B_p be a small ball, centered at $z \in F$, transverse to the trajectories. The trajectory leaves from a point y of B_p, comes back and intersects B_p again at a point $\varphi(y)$, if y is close enough to z; the function $y \to \varphi(y)$ is a homeomorphism from a neighborhood of z in B_p to a neighborhood of z (the homeomorphism is of class r if the given system is of class r). The holonomy of F is in this case a representation that sends a generator of $\pi_1(F, z) = Z$ to the germ of φ at z. The two results cited above are obvious in this case particularly.

[9] DZ: I do not know what exactly it means here by the right bracket of $[0, \infty]$, but I guess I will just leave it here.

[10] DZ: Seems that inside $|y| \leq 1/2$ when $p = 1$ it is the usual Reeb foliation.

2.2 RECALL ON THE GERMS OF FUNCTIONS

Let X and Y be topological spaces. Two continuous functions f and f' from a neighborhood of $x \in X$ to Y have the same germ at x if their restrictions to a given neighborhood of x coincide. Thus the *germ of* f *at* x is a set of all functions f' from the neighborhood of x to Y that are equal to f in a neighborhood of x (this neighborhood depends on f'). The point x is called a *source* of germ of f at x, and the point $y = f(x)$ is a *target*.

If g is a continuous function from a neighborhood of $f(x)$ to a topological space Z, the germ of gf at the point x depends only on the germs of f at x and g at y and is called the composition of these germs. So the germs of local homeomorphisms of class r of $\mathbb{R}^p$ at the origin that leaves 0 fixed, form a group.

2.3 THE SPACE OF DISTINGUISHED GERMS OF A FOLIATION

Let $\mathcal{F}$ be a foliated structure of class r on a manifold V. Consider the set $\tilde{V}$ of all the germs of distinguished functions of $\mathcal{F}$ at different points of V (we call the set of distinguished germs of $\mathcal{F}$); this set $\tilde{V}$ comes with a projection α onto V, which corresponds to each distinguished germ at the point x the point x itself. For each distinguished function f of $\mathcal{F}$, defined on an open set U of V, correspond a cover $\tilde{U}$ to U in $\tilde{V}$ formed by all the germs of f at points of U; the subsets $\tilde{U}$ of $\tilde{V}$ obtained in this manner form a basis of the topology of $\tilde{V}$. The projection α is locally a homeomorphism.

As the topology T of leaves of $\mathcal{F}$ on V is finer than the topology of V, there exists a unique topology $\tilde{T}$ on $\tilde{V}$ (also called the topology of leaves on $\tilde{V}$) such that α is again a local homeomorphism when $\tilde{V}$ and V are equipped with the topologies $\tilde{T}$ and T.

In general, $\tilde{V}$ with α is not a cover of V.

However, let us check the following remarkable fact.

Proposition 2.1. *The space* $\tilde{V}$ *of distinguished germs, with the projection*[11] α*, is a cover of* V*, when* $\tilde{V}$ *and* V *are equipped with the topology of leaves* $\tilde{T}$ *and* T*.*

Proof. Let $z \in V$ and f be a distinguished function defined in an open neighborhood U of z. The subset $U_0 = f^{-1}(x)$ of U, where $x = f(z)$, is an open set for the topology T. Each element of $\alpha^{-1}(U_0)$ is a germ at a point $z' \in U_0$ of a distinguished function of the form $h.f$, where h is a local homeomorphism of $\mathbb{R}^p$ of class r, defined on a neighborhood of x. The germs of $h.f$ at points of U_0 form a cover U_h of U_0 in $\tilde{V}$ and an open set for the topology $\tilde{T}$. The projection α maps homeomorphically U_h onto U_0. On the other hand, if h' is also a local homeomorphism of $\mathbb{R}^p$ and class r, defined in a neighborhood of x, then U_h and $U_{h'}$ only have points in common if h and h' have the same germ at x, in which case they coincide. Therefore $\alpha^{-1}(U_0)$ is a disjoint union of open sets of $\tilde{T}$ mapped homeomorphically by α onto U_0, this concludes the proof. ∎

2.4 THE HOLONOMY OF A LEAF

Let G be a group of germs of local homeomorphisms of $\mathbb{R}^p$ with source and target being the origin (cf. 2.2). *Given a leaf* F *of a foliated structure* $\mathcal{F}$ *on* V*, let us construct a representation* Φ *of* $\pi_1(F,z), z \in F$*, to* G*, defined up to inner automorphisms of* G*, and call it the holonomy of* F*.* It would be more correct to define the holonomy of F as an element of $H^1(F;G)$, the first cohomology group of F with coefficient G (cf. [Hae58]).

Φ will be determined unambiguously by the data of the germ $\tilde{z}$ at z of a distinguished function such that $f(z) = 0$. Let $c : [0,1] \to F$ be a loop in F based at z whose homotopy class gives an element $\gamma \in \pi_1(F,z)$; the loop c lifts to a unique path $\tilde{c}$, in the cover $\tilde{V}$ of V, starting from $\tilde{z}$. The end of $\tilde{c}$ is the germ $\tilde{z}'$ at z of a distinguished function f' such that $f'(z) = 0$. Thus there exists a unique element g of G such that $\tilde{z}' = g\tilde{z}$ (composition of germs, cf. 2.2); g only depends on γ. The correspondence

[11] DZ: The original sentence was "muni de la projection source α". What does this "source" mean?

associating to γ the element g is a homomorphism Φ from $\pi_1(F,z)$ into G (with the opposite convention to that if γ_1 and $\gamma_2 \in \pi_1(F,z)$, then $\gamma_2\gamma_1$ is a homotopy class of a path obtained by first travel along a path c_1 that represents γ_1 and then a path c_2 that represents γ_2).

If we had started from a distinguished germ $\tilde{z}'$ at z, the representation Φ constructed from $\tilde{z}'$ might be deduced from the construction starting from $\tilde{z}$ by composing it with the inner automorphism determined by g, where g is an element of G defined by $\tilde{z}' = g\tilde{z}$.

The holonomy group of F is the subgroup of G, defined up to conjugacy, the image of $\pi_1(F,z)$ by Φ.

Remark 2.2. The notion of holonomy can be extended without any change to the case of foliated Γ-structures (see remark of 1.2) that satisfies the condition of non-degeneracy as follows: given a distinguished function f, a point z of its domain and two elements h and h' of Γ defined at the point $f(z)$, if hf and $h'f$ have the same germ at z, then h and h' have the same germ at $f(z)$.

EXAMPLE. In the examples 1.5 and 1.7, the holonomy groups are all identity. In example 1.8, the holonomy group of a leaf $F(y)$ is isomorphic to the group of germs at y of elements of $\Phi(\Pi)$ that leaves y fixed. The fundamental group of $F(y)$ is isomorphic to subgroup Π_y of Π formed by the elements g such that $\Phi(g)y = y$. The holonomy of $F(y)$ is equivalent to the representation that associates to each element g of Π_y the germ of $\Phi(g)$ at y. In the specific example of 1.8, the holonomy group of a leaf is identity or of order 2 depending on if the leaf is an annulus or a Möbius band.

In general, if the holonomy group of a leaf F of a foliation of codimension 1 is finite, it is either the identity, or of order 2 depending on whether F is orientable or not.

For the classic foliation on $\mathbb{S}^3$, the holonomy of the compact leaf homeomorphic to a torus sends the generator represented by the meridian (resp. the parallel) to the germ of a homeomorphism at the origin of $\mathbb{R}$ that is identity on the negative half-line (resp. positive) and that is not the identity on the other half-line. This shows that the foliation is not analytic.

2.5 GEOMETRIC INTERPRETATION OF THE HOLONOMY

The following considerations should clarify the geometric meaning of the holonomy of a leaf.

Let C be a path in a leaf F and let T_0 and T_1 be the submanifolds transverse to $\mathcal{F}$ containing $z_0 = C(0)$ and $z_1 = C(1)$. For each neighborhood U of C in V, there exists a homeomorphism Φ_C of class r from a neighborhood of z_0 in T_0 to a neighborhood of z_1 in T_1 that satisfies the properties:

- *(i) If Φ_C is defined at $z \in T_0$, then $\Phi_C(z)$ belongs to the intersection of T_1 and the leaf passing through z of the foliation induced by $\mathcal{F}$ on U.*
- *(ii) The germ of Φ_C at z_0 neither depends on the U nor depends on the path C in the homotopy class.*
- *(iii) Suppose $z_0 = z_1$ and $T_0 = T_1$; let γ denote the homotopy class of the loop C in F; let $\bar{f}$ be the restriction to T_0 of a distinguished function f such that $f(z_0) = 0$. Then the germ of $\bar{f}\Phi_C\bar{f}^{-1}$ at 0 is the image of γ by the holonomy representation $\Phi : \pi_1(F,z_0) \to G$ (cf. 2.4).*

To construct Φ_C, consider a sequence of distinguished functions $f_i, i = 0,1,\dots,r$, defined on the open sets V_i and a sequence of increasing t_i of points on $[0,1]$, $t_0 = 0$ and $t_r = 1$, such that $C([t_k,t_{k+1}]) \subset V_k$. Let T^i be the submanifolds transverse to $\mathcal{F}$ containing $C(t_i)$ and such that $T^0 = T_0$ and $T^r = T_1$. We can suppose that $f_iC(t_i) = 0$ and that f_i is of the form πh_i^{-1}, where h_i is a local chart of $\mathcal{F}$ (cf. 1.3). For each $i < r$, there exists a homeomorphism Φ_i of class r from a neighborhood of $C(t_i)$ in T^i to a neighborhood of $C(t_{i+1})$ in T^{i+1} such that, if $z_{i+1} = \Phi_i(z_i)$, z_{i+1} sits in the leaf of $V_i \cap U$ passing through z_i .Then Φ_C is a composition of all the homeomorphisms $\Phi_0\Phi_1\dots\Phi_{r-1}$.

We can also construct Φ_C by constructing a continuous function Ψ of $B\times I$ in U, where B is in a neighborhood of z_0 in T_0, such that $\Psi(b,1)$ is contained in the leaf passing through b, $\Psi(z_0,t) = C(t)$, $\Psi(b,0) = b$ and $\Psi(b,1) \in T_1$.

The verification of the properties (ii) and (iii) follow immediately from 2.3 and 2.4.

Corollary 2.3. *Let F be a leaf whose holonomy group contains an element that is the germ at 0 of a local homeomorphism h of $\mathbb{R}^p$ such that, for a point $x \neq 0$ of $\mathbb{R}^p$, $h^m(x)$ is defined for each positive integer m and that $h^m(x)$ has 0 as a limit. There then exists a leaf, distinct from F if F is proper, whose closure contains F, and vice versa.*

The germ of h is the image by the holonomy Φ of F of an element $\gamma \in \pi_1(F, z)$. Let C be a path in F at z that represents γ and let T_0 be a submanifold transverse to the foliation and contain z. The local homeomorphism Φ_C and T constructed previously satisfies the same condition as h, namely there exists a point $y \in T_0$ such that $y^m = \Phi_C^m(t)$ is defined for each positive m, y^m tends to z. Then from (i), the points y^m belong to a same leaf whose closure contains z, and so is F according to 1.2.

2.6 THE STABILITY THEOREM

Theorem 2.4. *Each compact leaf F whose holonomy group is finite possesses a fundamental system of neighborhoods (open or compact) that is a union of compact leaves ([WR52], [ES56]).*

Note that the holonomy group of F is always finite if the fundamental group of F is finite.

Proof. Let $\mathcal{F}$ be a foliation on $\tilde{V}$, inverse image of $\mathcal{F}$ by $\alpha : \tilde{V} \to V$. Although $\tilde{V}$ is not separated (except in the case when analytic), the foliation $\tilde{\mathcal{F}}$ is in particular simple. In fact the function β from $\tilde{V}$ to $\mathbb{R}^p$ associating to each distinguished germ its target is a global distinguished function of $\tilde{F}$ (so that $\tilde{\mathcal{F}}$ is simple in the sense of 1.4)[12].

The topology of leaves of $\tilde{\mathcal{F}}$ is the topology $\tilde{T}$ defined in 2.3. According to the proposition of 2.3, each leaf F_0 of $\tilde{\mathcal{F}}$ is a covering of the leaf $\alpha(F_0)$ of $\mathcal{F}$, with the the topology of leaf. If the group of holonomy of F is finite, each leaf F_0 of $\tilde{\mathcal{F}}$ projecting to F is a finite-sheeted covering of F; therefore if F is compact, F_0 is as well. We thus remain to show that F_0 possesses a fundamental system of neighborhoods in $\tilde{V}$ that are saturated by compact leaves; the images by α of these neighborhoods will form the fundamental system that we seek.

Let Ω_0 be an open neighborhood of F_0 in V. Let $b_0 = \beta(F_0)$. Let W be a finite union of compact sets of $\tilde{V}$ such that W is a neighborhood of F_0 contained in Ω_0 and $W \cap \beta^{-1}(b_0) = F_0$. Let W_0 be the interior of W; the subset $D = W - W_0$ is a finite union of compacts, so $\beta(D)$ is a compact set that does not contain b_0. Let U be a neighborhood of b_0 that does not meet $\beta(D)$ and set $\Phi_0 = \beta^{-1}(U) \cap W_0$.

Φ_0 is a neighborhood of F_0 contained in Ω_0. It is saturated by compact leaves of $\tilde{\mathcal{F}}$. In fact, for each $z \in \Phi_0$ we have $W^0 \cap \beta^{-1}(b) = W \cap \beta^{-1}(b)$, where $b = \beta(z)$, since $b \notin \beta(W - W^0)$[13]. Thus $W^0 \cap \beta^{-1}(b)$ is an open set for the topology $\tilde{T}$ that is also closed[14] and compact, since it is a union of compact sets and that $\tilde{T}$ is separated; it is therefore a union of connected components of compact sets of $\beta^{-1}(b)$, so a union of compact leaves of $\tilde{\mathcal{F}}$. ∎

The differentiable case. Any proper leaf F of a differentiable $\mathcal{F}$ of class $r > 0$ in V admits a tubular neighborhood enjoying the following properties: U is given a projection q to F of class r such that $q(x) = x$ if $x \in F$ and that each fibre $q^{-1}(x)$ is a submanifold transverse to $\mathcal{F}$. According to the stability theorem, if the group of holonomy of a compact leaf F is finite, we can choose U as a union of compact leaves. We are then exactly in the situation of the example 1.8. The stability theorem can thus be stated in a more precise form:

Let F be a compact leaf with finite holonomy of a differentiable foliation. Then F admits a fundamental system of neighborhoods that are unions of compact leaves, these leaves are all diffeomorphic to covering spaces of F and their holonomy groups are isomorphic to a subgroup of the holonomy group of F.

[12]DZ: 1.5.

[13]DZ: W_0? $\beta^{-1}(U) \cap D = \varnothing$.

[14]DZ: Closed and open both by definition.

Remark 2.5. It results from the proof of Reeb (cf. [WR52], p. 121-4) that in the topological case, the leaves neighboring F have fundamental groups which are isomorphic to subgroups of the fundamental group of F and have also holonomy groups isomorphic to subgroups of holonomy group of F.

2.7 THE HOLONOMY CHARACTERIZES THE FOLIATED NEIGHBORHOOD OF A LEAF

Let us place ourselves in the category of differentiable foliations of class $r > 0$ and codimension p defined on a paracompact manifold. G denotes the group of germs at the origin 0 of homeomorphisms of class r of $\mathbb{R}^p$ leaving 0 fixed.

Theorem 2.6. *a) Existence. Let F be a connected paracompact manifold of class r and let Φ be a representation from $\pi_1(F, z)$ into G. There exists then a manifold V with a foliation of class r and codimension p, possessing a proper leaf isomorphic to F and whose holonomy is Φ.*

b) Uniqueness. Let $\mathcal{F}_i (i = 1, 2)$ be two foliations of class r and codimension p on the paracompact manifolds V_i. Suppose that there exists a homeomorphism ψ of class r from a proper leaf F_1 of $\mathcal{F}_1$ to a proper leaf F_2 of $\mathcal{F}_2$ so that the diagram

$$\begin{array}{ccc} \pi_1(F_1, z_1) & \xrightarrow{\psi^*} & \pi_1(F_2, z_2) \\ & {\scriptstyle \Phi_1} \searrow \quad \swarrow {\scriptstyle \Phi_2} & \\ & G & \end{array}$$

is commutative, where Φ_i is the holonomy of F_i, $z_2 = \psi(z_1)$. There exists then a homeomorphism Ψ of class r from a neighborhood U_1 of F_1 to a neighborhood U_2 of F_2 extending ψ that is an isomorphism of foliations induced on U_i by $\mathcal{F}_i$.

The proof of this theorem is elementary and consists essentially regluing the pieces properly (cf. [Hae58] p. 298-301 and 303-304).

The theorem is true in the differentiable case. It is also true in the analytic case (as a consequence of theorem of GRAUERT and MORREY according to whom every analytic manifold can be embedded analytically into $\mathbb{R}^N$). On the other hand we do not know if b) is true in the topological case.

With theorems a) and b) we can deduce, using the following elementary lemma, the stability theorem in the differentiable case in its strongest form (namely that F possesses a foliated neighborhood determined as in 1.8, by a representation Φ from $\pi_1(F, x)$ into the homeomorphism group of class r of a ball B, the map associating to an element of H its germ at 0 is bijective).[15]

Lemma 2.7. *Let H_0 be a finite subgroup of the group of germs of homeomorphisms at a point z of a topological space V. There exists then an arbitrary small neighborhood W of z and a group H of homeomorphisms of W such that the map associating to each element of H its germ at z is an isomorphism from H to H_0.*

2.8 THE INFINITESIMAL HOLONOMY

Let G_r be a group of jets of order r at the origin of homeomorphisms of class r of $\mathbb{R}^p$ leaving 0 fixed; the group G_r is the quotient of G by the subgroup formed by elements of G that are tangent to the identity at 0 up to the order r[16].

If $\mathcal{F}$ is a foliation of order $r' \geq r > 0$, the infinitesimal holonomy of order r of a leaf F is the composition $\Phi_r : \pi_1(F, z) \to G_r$ of the holonomy Φ with the natural homomorphism of G onto G_r. The group of infinitesimal holonomy of order r of F is the image of Φ_r.

The consideration of this group allows us to decide in certain particular cases if the holonomy group is finite, which then makes it possible to apply the stability theorem (cf. G. REEB, Remarques sur les structures feuilletées, Bull. Soc. Math. France, 87 (1959), p. 445-450)).

[15] DZ: The bundle is formed by the normal neighborhood of $\tilde{F}$ lifted as in the proof of Theorem 2.4, which is a product, modulo the diagonal action of the kernel of the holonomy representation of F.

[16] DZ: In other words (from Wikipedia), two functions are equivalent if they have the same value at 0 and all the partial derivatives agree at 0 up to (including) the r-th order derivatives.

The following theorem is due to C. Ehresmann. (cf. [Ehr52]).

Theorem 2.8. *Let F be a closed submanifold of class $r > 0$ of a manifold V of class r. For F to admit a neighborhood with a foliation of class r of which F is a leaf, it is necessary and sufficient that the normal bundle of F in V admits a discrete structural group.*

Proof. The condition is necessary. Let f_i $(i \in I)$ be a family of distinguished functions such that $f_i^{-1}(0) = U_i^0$ forms an open cover of F. Let E^p be the space of tangent vectors of $\mathbb{R}^p$ at 0. The differential of each f_i defines a fibred function f_i^ν of the fibre space of the normal vectors to F, restricted to U_i^0, on E^p. According to the cocycle condition[17] of distinguished functions, at $z \in U_i^0 \cap U_j^0$, we can get from f_i^ν to f_j^ν by composing f_i^ν with a linear automorphism of E^p which is a locally constant function at z. Therefore the functions f_i^ν define *on the fibre of normal vectors to F a structure of fibre space with discrete structural group which is perecisely the one that is given by the infinitesimal holonomy $\Phi_1 : \pi_1(F, z) \to G_1$ of order 1 of F.*

The condition is sufficient. If the normal bundle N of F admits a restriction of its structure group to a discrete group determined by a homomorphism $\Phi_1 : \pi_1(F, z) \to G_1$, we can apply the construction of 1.8. We obtain thus on N a foliation of which F, identified with the zero section, is a leaf. We can identify a neighborhood of F in N with a neighborhood of F in V and then obtain a foliation in a neighborhood of F such that the infinitesimal holonomy of order 1 of F is given by Φ_1. ∎

For example, a complex projective line d in the complex projective plane cannot be a leaf of a differentiable foliation defined in the neighborhood of d.

3 TOPOLOGICAL AND DIFFERENTIABLE FOLIATIONS OF CODIMENSION 1

In this whole section, V denotes a manifold with a countable basis whose first rational Betti number is finite. According to 1.3, each closed leaf is proper.

3.1

Proposition 3.1. *Let $\mathcal{F}$ be a topological foliation on V of codimension 1 and orientable. Through any point z of V passes a transverse curve to $\mathcal{F}$ intersecting each closed leaf at at most one point.*

The proof relies on two lemmas ([Hae56]).

Lemma 3.2. *If the first rational Betti number of a connected V is equal to p, then there exist m closed leaves $F_1, \dots, F_m$, where $0 \le m \le p$, such that the complement V_m of $\cup_{i=1}^m F_i$ is connected and that if F_{m+1} is another closed leaf, the complement V_{m+1} of $U_{i=1}^{m+1} F_i$ has two connected components.*

Proof. Let $F_1, \dots, F_q$, be q distinct leaves such that the complement V_q of the union Φ^q of F_i, $0 \le i \le q$, is connected. The exact sequence of cohomology with compact supports in V relatively to closed subspace Φ^q gives:

$$H^{n-1}(V) \xrightarrow{i} H^{n-1}(\Phi^q) \xrightarrow{\delta} H^n(V_q) \xrightarrow{\alpha} H^n(V) \longrightarrow H^n(\Phi^q),$$

the coefficients being the local system of twisted integers by the orientations of V, V_q and Φ_q.

By Poincaré duality, $H^n(V_q)$ and $H^n(V)$ are isomorphic respectively to $H_0(V_q)$ and $H_0(V)$, with integer coefficients, which is said to be $\mathbb{Z}$. The groups $H^{n-1}(V)$ and $H^{n-1}(\Phi^q)$ are isomorphic to $H_1(V)$ and to $H_0(\Phi^q)$ respectively. So they are of rank p and q respectively. As α is an isomorphism, i is surjective, $q \le p$. Thus there exist an integer $m \le p$ and m leaves $F_1, \dots, F_m$ that satisfy the statement of the lemma. ∎

[17] DZ: The original was "condition liant".

Lemma 3.3. *If the complementary subspace of each closed leaf has two distinct connected components, a transversal cannot intersect a closed leaf at more than one points.*

Proof. Note that first of all as a result, each closed leaf F is orientable and that if a transverse T intersects F in z, the points of T on either side of z and very close to z, are sitting in distinct connected components of the complement of F. If T intersects F elsewhere, there exists a point $z_1 \in F \cap T$ such that the segment $T(z, z_1)$ with endpoints z and z_1 on T, do not meet F except for at z and z_1. Each point z_2 of T near z_1 but not on $T(z, z_1)$ is in the other connected component of the complement of F while we let z_0 denote a point inside $T(z, z_1)$. Now since F is orientable, each leaf intersecting $T(z, z_1)$, in the point z_0 close enough to z, will intersect T at a point z_2 sitting outside $T(z, z_1)$ and close to z_1[18], but this countradicts to the fact that z_0 and z_1[19] are in the distinct connected components. ∎

Proof of the proposition. Let $F_1, \dots, F_m$ be closed leaves satisfying the statement of Lemma 3.2. By Lemma 3.3, the theorem is true at each point of the complement of the F_i's.[20] Now let T be a transverse curve intersecting F_i at z only, which does not intersect the other F_j's, $j \neq i$. If T intersects another closed leaf F at two points z_0 and z_1, according to Lemma 3.3[21], these points locate on either side of z on T and T cannot intersect F elsewhere. The complement of F in V'_m = (the complement in V of the union of the F_i's) has two connected components V_m^+ and V_m^- and the foliation $\mathcal{F}$ is transversely orientable. So two points sitting in the interior of T_0 of the segment $T(z_0, z_1)$ and on different sides of z are contained in the distinct connected components V_m^+ and V_m^-. Thus T_0 cannot meet a leaf at more than two points and satisfies the statement of the theorem. ∎

3.2 THE SET OF CLOSED LEAVES

Theorem 3.4. *In a foliation of codimension* 1 *on* V*, the closure of a union of closed leaves is also a union of closed leaves.*

Proof. It suffices to prove in the case where the foliation is orientable, if not we can pass to a double cover of V (cf. 1.4). Let F be a leaf in the closure of the union of closed leaves. To show that F is closed, it suffices to show that, at each $z \in V$, there passes a transversal intersecting F at a single point[22]; by Theorem 3.1, we already know that, at z, there passes a transversal intersecting any closed leaf in at most one point; if this transversal intersects F at two points, it would also intersect a closed leaf sufficiently close to F at two points, which contradicts to our hypothesis. ∎

Here is another consequence of Theorem 3.4.

Theorem 3.5. *Let* V *be a connected compact manifold endowed with a foliated structure of codimension* 1 *and transversely analytic (cf. 1.1). Then either all the leaves are compact, or there is at most a finite number of compact leaves ([Hae58]).*

Proof. From Theorem 3.4, the set of all compact leaves is a compact set K; the compact leaves whose holonomy groups are finite form an open set according to the stability theorem. It is also a closed set, because the compact leaves whose holonomy group is infinite are isolated in K (cf. Lemma 2, p. 328 of [Hae58]; this is where the analyticity comes into play).

So if a compact leaf has a finite holonomy group, it is the same for all leaves. Otherwise, there are at most a finite number of compact leaves. ∎

[18] DZ: ...because of the orientation.

[19] DZ: z_2.

[20] DZ: For closed leaves also in the complement.

[21] DZ: This case only happens when adding F_i the complement of F has only 1 connected component.

[22] DZ: I guess here V is assumed to be closed, so if F is not closed...

3.3 THE CASE WHERE ALL LEAVES ARE CLOSED

Theorem 3.6. *Let $\mathcal{F}$ be a foliation of codimension* 1 *which is orientable and all the leaves of which are closed. Then the leaf space is a manifold of dimension* 1*, in general not separated, and orientable.*

Proof. According to Theorem 3.1, at any point in V there passes a transversal intersecting each leaf in at most one point. This means that the leaf space is locally homeomorphic to a manifold of dimension one.

When $\mathcal{F}$ is not orientable, the leaf space is a manifold of dimension 1 with a finite number of boundary points.

When all the leaves are compact, the leaf space is a *separated* manifold of dimension one; in fact each leaf has a finite holonomy group (cf. corollary 2.5); according to the stability theorem, two distinct leaves have disjoint neighborhoods which are the union of leaves. If $\mathcal{F}$ is orientable, then the leaf space is homeomorphic to a circle or to a line depending on whether V is compact or not.

If moreover $\mathcal{F}$ is of class $r > 0$, the leaves are the fibers of a fibration of class r of V (cf. 2.6). If $\mathcal{F}$ is non-orientable, the leaf space is homeomorphic to a closed or semi-closed segment depending on whether V is compact or not. ∎

3.4 EXISTENCE OF A FIRST INTEGRAL

Let $\mathcal{F}$ be a foliation of codimension 1 and of class r on a manifold V. A first integral f of class $r' \leq r$ of $\mathcal{F}$ is a global distinguished funcion of the underlying foliation $\mathcal{F}$ of class r'. In other words, f is a map of class r' from V to $\mathbb{R}$, constant on the leaves of $\mathcal{F}$, and such that its restriction to any transverse curve of class r is a homeomorphism of class r' in $\mathbb{R}$.

The existence of the first integral f implies that the leaf space V_0 of $\mathcal{F}$ is a manifold, in general not separated, of dimension 1 and orientable (in particular all the leaves are closed); moreover there exists a map f_0 from V_0 to $\mathbb{R}$ defined by the condition $f_0\pi = f$, where π is the natural projection of V to V_0; f_0 is locally a homeomorphism of class r' from V_0 to $\mathbb{R}$.

$$\begin{array}{ccc} V & & \\ \downarrow & \searrow^{f} & \\ V_0 & \xrightarrow[f_0]{} & \mathbb{R} \end{array}$$

Conversely, if the leaf space is a manifold V_0 of dimension 1 and if f_0 is a function on V_0 that is locally a homeomorphism of class r', then $f = f_0\pi$ is a first integral of class r'.

The problem of the construction of f is therefore reduced to that of the construction of f_0.

Theorem 3.7. *Let V be a manifold (with a countable basis) and whose first rational Betti number is zero. Let $\mathcal{F}$ be a foliation of codimension* 1 *on V, orientable, of class $r < \omega$ and all the leaves of which are closed. Then $\mathcal{F}$ admits a continuous first integral. The foliated structure induced on a relatively compact open set V', admits a first integral of class r.*

In this statement, $r \neq \omega$ (see the remark of 4.4). This theorem is a generalization of a theorem of Kamke ([HR57] and [Hae56]).

From 3.3, the leaf space is a non-separated manifold V_0 of dimension 1 with a countable basis; moreover (cf. 3.1, Lemma 3.2), the complement of any point of V_0 has two connected components. As a result (cf. [HR57], p. 113-114) there exists a function f_0 from V_0 to $\mathbb{R}$ which is locally a homeomorphism.

If $\mathcal{F}$ is of class $r > 0$, there is usually no global first integral of class r, as already shown by the simplest foliated structures of the plane (cf. [HR57]). This is because V_0 generally does not satisfy the following condition.

Let us say that a manifold V_0 of dimension 1, not separated, is endowed with a *regular* differentiable structure of class r, if for any function f of class r defined on a neighborhood of $x \in V_0$, there exists a function f' of class r defined on V_0 such that f and f' coincide on a neighborhood of x.

In [HR57], p. 117-8, it is shown that if a manifold V_0 satisfies the previous condition, and the complement of any point of V_0 has two connected components, then there exists a function f_0 from V_0 to $\mathbb{R}$ which is locally a homeomorphism of class r. The theorem will therefore be a consequence of the following lemma.

Lemma 3.8. *If the leaf space V_0 of a foliation $\mathcal{F}$ of class r and codimension 1 on V is a manifold (possibly non-separated), then the leaf space V_0' of a foliated structure $\mathcal{F}$ induced on a relatively compact open V' of V is a manifold of a regular differentiable structure of class r.*

Proof. Let π and π' be the natural projections of V and V' to V_0 and V_0' respectively. The injection i of V' into V induces by passing to the quotients a function ψ of V_0' in V_0 such that $\pi i = \psi\pi'$ which is locally a homeomorphism of class r.

Let x' be a point of V_0' and let U be an open separated neighborhood of $x = \psi(x')$; let F be a leaf $\pi^{-1}(x)$. We can construct in $\pi^{-1}(U)$ a compact neighborhood K of the intersection of F with the closure $\bar{V}'$ in V. The image under π of the intersection of the boundary ∂K of K with $\bar{V}'$ is a compact set contained in U and not containing x. So let L be a compact neighborhood, homeomorphic to an interval, of x contained in $\pi(K)$ and not meeting $\pi(\partial K \cap \bar{V}')$. Then $W = \pi^{-1}(L) \cap K \cap V'$ is a closed subset of V' which is a union of the leaves of $\mathcal{F}'$. Indeed, for all $y \in L$, $\pi^{-1}(y) \cap K \cap V'$ is both open and closed in $\pi^{-1}(y) \cap V'$.

Let f be a function of class r defined on the neighborhood of x'; we can construct a function g of class r on L, vanishing on the neighborhood of the boundary of L, and such that $g\psi$ is equal to f in the neighborhood of x'. Then the function on V' equal to 0 outside W and to $g\pi$ on W is of class r; since it is constant on the leaves of $\mathcal{F}'$, it defines by passing to the quotient a function f' of class r on V_0' which coincides with f in the neighborhood of x'. ∎

3.5 THE GLOBAL STABILITY THEOREM OF REEB

We cannot end this section without citing one of the most remarkable properties of codimension 1 foliations demonstrated by Reeb ([WR52], pp. 134-140).

Theorem 3.9. *Let V be a connected compact manifold endowed with a foliation of codimension 1. If a leaf is compact and has a finite fundamental group, then all leaves are compact and have finite fundamental groups.*

We can refer to 3.3 for more precise consequences. Example 1.10 shows that the theorem is no longer true in codimension greater than 1.

We will limit ourselves to sketching a proof in the differentiable case.

By stability theorem 2.4, compact leaves with finite fundamental group form an open set U in V. Since V is connected, it suffices to show that a connected component U_0 of U is also closed. Any leaf F adhering to U_0 is also compact according to 3.2 (for a more direct reasoning, cf. Reeb [WR52], p. 136). It will therefore suffice to verify that its fundamental group is also finite.

Without loss of generality, we can assume the foliation is orientable (cf. 1.4). Let W be a tubular neighborhood of F with its projection $q : W \to F$, which is small enough so that the fibers are transverse to the leaves (cf. 2.6). Any leaf of U_0 close enough to F will be contained in W and intersect each fiber of W at a single point; it will therefore be diffeomorphic to F. So the fundamental group of F will also be finite. ∎

Here is an interesting application of this theorem also due to Reeb (unpublished).

Theorem 3.10. *Let V be a simply connected compact manifold, having a nonempty boundary that is connected and simply connected. There cannot exist on V a foliation of codimension 1 such that the boundary of V is a leaf.*

If there was such a foliation, all the leaves would be compact and simply connected. The leaf space would be a compact manifold of dimension 1 having boundary reduced to a point (corresponding to the boundary of V), which is impossible.

For example, the manifold $V = \mathbb{S}^p \times \mathbb{D}^{n-p}$, product of a sphere of dimension p and a disk of dimension $n-p$, is simply connected and as well as its boundary for $p > 1$ and $n-p-1 > 1$. Although there is no codimension 1 foliation on V whose boundary is a leaf, the field of $(n-1)$-planes tangent to the boundary can be extended to the interior of V if the Euler characteristic of V is zero, therefore if p is odd.

For other applications of the global stability theorem, see Reeb [WR52], p. 147.

4 ANALYTIC FOLIATION OF CODIMENSION ONE

4.1

Let V be a manifold endowed with an analytic foliation $\mathcal{F}$ of codimension 1. A closed transverse curve to $\mathcal{F}$ is a continuous map τ of the circle $\mathbb{S}^1$ in V such that, for any distinguished function f_i of $\mathcal{F}$, the function $f_i\tau$ is locally a homeomorphism of $\mathbb{S}^1$ to $\mathbb{R}$.

Lemma 4.1 (Fundamental Lemma)**.** *A closed curve transverse to an analytic foliation of codimension 1 on V represents an element of infinite order of the fundamental group of V.*

Note that a closed transverse curve could be homologous to zero.

This lemma follows immediately from the following property of differentiable foliations.

4.2

Proposition 4.2. *Let $\mathcal{F}$ be a foliation of codimension 1 and of class 2 on a manifold V. Suppose that there exists a closed transversal to $\mathcal{F}$ homotopic to a constant loop. Then there is a loop on a leaf F such that the germ of homeomorphism of $\mathbb{R}$ at 0 which corresponds to it by the holonomy of F is not the identity, but it is the germ of a homeomorphism which is identity on $(-\infty, 0]$ or on $[0, \infty)$.*

Proof. The closed transversal is a function τ from $\mathbb{S}^1$ to V that can be assumed differentiable of class 2 and such that, for each distinguished function f_i, the function $f_i\tau$ is locally a homemorphism of class 2 from $\mathbb{S}^1$ into $\mathbb{R}$. As τ is homotopic to a constant map, it is possible to extend it to a function $\varphi : \mathbb{D} \to V$ of class 2 of the disk $\mathbb{D}$ bounded by $\mathbb{S}^1$ in the plane.

By applying a well-known theorem of Morse (cf. [ES56], p. 316-17), it is possible to choose φ so that, for any distinguished function f_i, $f_i\varphi$ is a non-degenerate number-valued function; this means that at each of its singular points (points where the first partial derivatives are zero), the matrix of second partial derivatives is non-singular. We therefore have on $\mathbb{D}$ a finite number of singular points for functions $f_i\varphi$ which are either of the maximum or minimum type, or of the saddle point type. We can suppose moreover (cf. [Hae58], p. 318) that the images by φ of two distinct singular points do not locate on the same leaf in a neighborhood of $\varphi(\mathbb{D})$.

We can notice that the functions $f_i\varphi$ are distinguished functions of a foliated Γ-structure on $\mathbb{D}$, where Γ is the pseudogroup of analytic local homeomorphisms of $\mathbb{R}$ (cf. 1.2). The leaves of this structure are curves (which may contain a singular point) which are the connected components of the intersections of $\mathbb{D}$ with the leaves of $\mathcal{F}$. We will consider them as trajectories (or union of trajectories if they contain a singular point) of a vector field on $\mathbb{D}$.

We can in fact construct a vector field X on $\mathbb{D}$ of class 1, which only vanishes at the singular points of the distinguished functions $\varphi_i = f_i\varphi$, and the differential of each φ_i maps X to the zero field of $\mathbb{R}$[23]. This is possible because the foliated structure on $\mathbb{D}$ is orientable (indeed, at any point we have exactly two germs of transverse orientation, and $\mathbb{D}$ is simply connected). We can therefore use the results of the classical theory of curves defined by differential equations. In Poincaré's terminology [Poi85], the singular points x_i of the field X are centers (if a distinguished function reaches at x_i a maximum or a minimum) or collars (a saddle point); there are at most 4 trajectories which end in a collar; moreover, two distinct collars are not connected by a trajectory. We do not have foci or nodes. The circle $\mathbb{S}^1$ is a cycle without contact, that is, a closed curve transverse to the trajectories.

Let L be the set of limit cycles of X on $\mathbb{D}$. More precisely, an element of L is a closed curve l in $\mathbb{D}$ which is either a closed trajectory of X, or the union of a trajectory of X and a saddle point if this trajectory starts and ends at this point; furthermore the image under φ of l must be a curve on a leaf F such that the element of the holonomy group corresponding to it is non-trival[24].

Note that first L is non-empty. In fact, there is only a finite number of trajectories which end at a singular point (saddle point); in general therefore, according to the Poincaré-Bendixon theorem (cf. [CL55]), a trajectory which intersects the boundary of $\mathbb{D}$ has as limit set either a closed trajectory C which is a limit cycle, or a limit polycycle which is the union $C_1 \cup C_2$ of a saddle point and two trajectories starting from this point and ending there. The element of the holonomy group corresponding to $\varphi(C)$ or to the loop obtained by traversing $\varphi(C_1)$ and then $\varphi(C_2)$ is not trivial (cf. 2.5); therefore at least one of those which corresponds to loops $\varphi(C_1)$ or $\varphi(C_2)$ is non-trivial.

The set L is partially ordered: if $l_1, l_2 \in L$, l_2 is less than l_1 if l_2 is located inside the domain bounded by l_1. Moreover, this ordered set is inductive. Let in fact L_0 be an infinite subset of L totally ordered. The elements of L_0 have as their limit either a closed traiectory C, or the union $C_1 \cup C_2$ of two trajectories and a saddle point from which they originate and where they end. The element of the holonomy group corresponding to $\varphi(C)$ is non-trivial, since it is the germ at 0 of a local homeomorphism of $\mathbb{R}$ which is not the identity in the neighborhood of a series of points tending towards 0 (these points correspond to the limit cycles which tend towards 0, cf. 2.5). The same reasoning shows that the element of the holonomy group corresponding to the loop obtained by traversing $\varphi(C_1)$ then $\varphi(C_2)$ is not trivial; it is therefore the same for the element corresponding to one of the loops $\varphi(C_1)$ or $\varphi(C_2)$.

Let l be a minimal element of L (such an element exists according to Zorn's theorem). All trajectories located in the interior of the domain $\mathbb{D}_l$ bounded by l are closed; if this was not the case, the Poincaré-Bendixon theorem would imply, as before, the existence of a limit cycle in $\mathbb{D}_l$, which would contradict the fact that l is minimal. The element of the holonomy group corresponding to $\varphi(l)$ is therefore the germ at 0 of a local homeomorphism of $\mathbb{R}$ which is the identity on one of the half-lines $(-\infty, 0]$ or $[0, \infty)$ (cf. 2.5). ∎

4.3 NON-EXISTENCE OF THE ANALYTIC FOLIATION

Theorem 4.3. *A compact analytic manifold whose fundamental group does not contain an element of finite order cannot support an analytic foliation of codimension* 1*.*

For any foliation of codimension 1 on a compact manifold V, there exists a closed transversal (for more details, cf. [Hae58] p. 324, corollary). According to Lemma 4.1 this transversal represents an element of infinite order of $\pi_1(V)$.

For other consequences of the Fundamental Lemma (existence of a compact leaf, ...), cf. [Hae58], p. 324, propos. 2 and p. 326-7, theorem 3.

[23] DZ: Meaning each trajectory lies in a leaf.

[24] DZ: There is local homeomorphism h of $\mathbb{R}$ at $\varphi(l)$ fixing the origin such that if x is close to 0 then $h^m(x) \to 0$.

4.4 EXISTENCE OF A GLOBAL FIRST INTEGRAL

Theorem 4.4. *Let V be an analytic manifold with a countable basis whose fundamental group contains only elements of finite order. If $\mathcal{F}$ is an analytic foliation of codimension 1 on V, every leaf is closed. If $\mathcal{F}$ is orientable, $\mathcal{F}$ admits a continuous global first integral; the restriction of $\mathcal{F}$ to a relatively compact open space admits a first integral of class ∞.*

To show that each leaf is closed, we can assume that $\mathcal{F}$ is orientable by passing if necessary to a double cover of V. However, for any orientable foliation of codimension 1 on V, the existence of a transverse curve meeting a leaf at two points results in the existence of a closed transversal (for more details, cf. [Hae58] p. 322, last §).

According to the Fundamental Lemma, therefore, a transverse curve cannot meet a leaf at more than one point, since $\pi_1(V)$ only has elements of finite order. We can therefore apply the considerations of 3.4.

Remark 4.5. In general, there is no such analytic first integral. It is easy to construct an example of foliation of $\mathbb{R}^n$ such that the leaf space V_0 of the foliation induced on a ball is a manifold obtained in the following way: we take two straight lines and we glue them along their negative parts using the homeomorphism $h(t) = t - t^2, t < 0$. There is no global analytic function on V_0 whose derivative is everywhere $\neq 0$.

4.5 ANALYTIC FOLIATIONS WITH SINGULARITIES

Let F be a real analytic manifold. An analytic foliation $\mathcal{F}$ of codimension 1 on V, with singularities, is defined as previously by a maximal set of analytic functions f_i (the distinguished finctions of $\mathcal{F}$) defined on open sets U_i forming an open cover of V and satisfying the condition: for all $x \in U_i \cap U_j$, there exists a local analytic homeomorphism h_{ji}^∞ of $\mathbb{R}$ such that $f_j = h_{ji}^\infty f_i$ in the neighborhood of x. We no longer assume this time that the functions f_i are of rank 1.

The following proposition makes it possible to extend to analytic foliations of codimension 1 with singularities most of the properties of non-singular structures.

Proposition 4.6. *Let $\mathcal{F}$ be an analytic foliation of codimension 1 with singularities on a paracompact manifold V. There exists then a paracompact analytic manifold, endowed with an analytic foliation $\mathcal{F}'$ of codimension 1 without singularities, and an analytic embedding φ from V into V' that is a homotopy equivalence and such that $\mathcal{F}$ is the inverse image of $\mathcal{F}'$ of φ.*

This means that the mappings obtained by composing φ with the distinguished functions of $\mathcal{F}'$ are distinguished functions of $\mathcal{F}$. This proposition is a special case of proposition 1, p. 314 of [Hae58].

Proof. If h is an analytic homeomorphism from an open connected set U of $\mathbb{R}$ to an open set of $\mathbb{R}$, we will denote by $\bar{h}$ the (unique) homeomorphism which can be defined on the largest interval containing U and which coincides with h on U.

If V is connected and if a distinguished function is constant, then all of them are; we can take for V' the product of V by $\mathbb{R}$. Let us therefore assume that the distinguished functions are not constant.

Let $(f_i)_{i\in I}$ be a family of distinguished functions defined on open sets U_i forming an open cover of V. In the disjoint union E of $U_i \times \mathbb{R}, i \in I$, the relation $(x_i, t_i) \sim (x_j, t_j)$ if and only if $x_i = x_j = x$ and $t_j = \bar{h}_{ji}^\infty t_i$, is an equivalence relation. It is indeed reflexive and symmetric because $\bar{h}_{ii}^\infty =$ identity of $\mathbb{R}$ and $\bar{h}_{ij}^\infty = (\bar{h}_{ji}^\infty)^{-1}$. It is also transitive; we have in fact $\bar{h}_{ki}^\infty f_i(x) = \bar{h}_{kj}^\infty \bar{h}_{ji}^\infty f_i(x)$, and the source of $\bar{h}_{kj}^\infty \bar{h}_{ji}^\infty$ is connected as an intersection of two intervals; thus this homeomorphism is a restriction of $\bar{h}_{ki}^\infty$.

Let V'' then be the quotient space of E by this equivalence relation. The canonical projection g_i from $U_i \times \mathbb{R}$ to V'' is a homeomorphism on an open set and the maps $g_j^{-1} g_i$ are analytic, because $g_j^{-1} g_i(x,t) = (x, \bar{h}_{ji}^\infty(t))$ and $\bar{h}_{ji}^\infty$ depends only on the connected component of $U_i \cap U_j$ containing x. So V'' is equipped with a structure of real analytic manifold, but V'' is not separated. The maps $f_i'' =$ the composition of g_i^{-1} with the natural projection of $U_i \times \mathbb{R}$ onto $\mathbb{R}$, are the distinguished functions of an analytic foliation $\mathcal{F}''$ of codimension 1 without singularity on V''.

The graphs $(x, f_i(x))$ of distinguished functions f_i of $\mathcal{F}$ give an embedding φ of V into V'' defined by $\varphi(x) = g_i(x, f_i(x))$, if $x \in U_i$. Note that $\varphi f_i = f_i''$, therefore $\mathcal{F}$ is the inverse image of $\mathcal{F}''$ of φ.

Finally, we can construct (for more details, see remark p. 315 of [Hae58]) a separated open neighborhood V' of $\varphi(V)$ in V'' which can deformation retract to $\varphi(V)$; $\mathcal{F}'$ will then be the foliation induced by $\mathcal{F}''$ on V. ∎

Corollary 4.7. *The Fundamental Lemma 4.1 as well as Theorem 4.4 are also valid for analytic foliations of codimension* 1 *with singularities.*

Theorem 4.3 implies that any analytic foliation of codimension 1 on a manifold V whose fundamental group is finite admits singularities. Note that there is always such a foliation, for example the one that is determined by a single distinguished function, which would be a non-constant analytic function on V.

4.6 APPLICATION TO THE COMPLETELY INTEGRABLE ANALYTIC PFAFF FORMS

On an analytic manifold V, let α be an analytic and completely integrable Pfaff form (1-form), that is, $d\alpha \wedge \alpha = 0$. In an open set U of V, a *first integral of class* $r > 0$ of α is a function f of class r in U such that $df = g\alpha$, where g is a function of class r different from zero in U (called an *integrating factor of class* r).

In the neighborhood of a point x which is not a singular point of α (i.e., a point where α does not vanish), there always exists an analytic first integral; moreover, if f and f' are two analytic first integrals defined in the neighborhood of x, they are related by an invertible analytic relation: there exists an analytic function h on an open set U to $\mathbb{R}$ whose derivative is everywhere $\neq 0$, that is, an analytic homeomorphism h, such that $f' = hf$ in the neighborhood of x.

In general, we will say that a *family of first integrals* f_i of α, defined on open sets U_i, forms a *coherent system*, if the U_i form an open cover of V and if, for all $x \in U_i \cap U_j$, there exists an analytic local homeomorphism $\bar{h}_{ji}^{\infty}$ of $\mathbb{R}$ such that $f_j = \bar{h}_{ji}^{\infty} f_i$ in the neighborhood of x. In other words, the first integrals f_i are the distinguished functions of an analytic foliation $\mathcal{F}$ of codimension 1 on V (with singularities in general). Note that $\mathcal{F}$ is orientable.

Conversely, the distinguished functions f_i of an analytic oriented foliation $\mathcal{F}$ of codimension 1 form a coherent system of first integrals of a 1-form α. In fact, if $f_j = h_{ji}^{\infty} f_i$, let g_{ji} be the derivative of h_{ji}^{∞} with respect to the natural coordinate of $\mathbb{R}$ taken at the point $f_i(x)$; it is a strictly positive analytic function defined on $U_i \cap U_j$. As $g_{ki} = g_{kj} g_{ji}$ on $U_i \cap U_j \cap U_k$, $\{g_{ji}\}$ is a 1-cocycle which determines an element of $H^1(V, \sigma^+)$, where σ^+ is the bundle of germs of positive analytic functions on V. But $H^1(V, \sigma^+)$ is always zero (cf. H. Cartan, Bull. Soc. Math. France, 85 (1957), p. 77-99). There are therefore strictly positive analytic functions g_i defined on an open cover U_i' finer than U_i such that $g_{ji} = g_j / g_i$ on $U_i' \cap U_j'$. Then the 1-form α will be equal to df_i / g_i on U_i'.

In summary, there is a one-to-one correspondence between transversely orientable analytic foliation of codimension 1 and a (maximal) coherent system of first integrals of Pfaff forms.

From what we recalled earlier, a completely integrable Pfaff form without singular points admits a coherent (and unique maximal) system of analytic first integrals. G. Reeb has shown ([WR52], p. 148-154) that the same is true if $\dim V > 2$ and if the form α completely integrable has only singular points where the determinant of the first partial derivatives of the coefficients of α is $\neq 0$.

From 4.5, a Pfaff form α on V admits a coherent system of analytic first integrals if and only if there exists an analytic embedding φ of V into an analytic manifold V' and an analytic Pfaff form α' on V' completely integrable without singular point such that $\alpha = \varphi^* \alpha'$.

4.7 finishing off (donne le?)

Theorem 4.8. *Let V be a connected real analytic manifold whose fundamental group does not have an element of finite order. A Pfaff form on V that admits a coherent system of analytic first integrals, admits an infinitely differentiable first integral on every open set relatively compact of V.*

Using the remark of 4.4, we can construct Pfaff form on the sphere $\mathbb{S}^2$ with 4 singular points, which admits a coherent system of analytic first integrals, but which does not admit global analytic first integrals.

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