An Example of a Transitive Anosov flow Transverse to a Torus and Not Conjugate to a Suspension

Christian Bonatti and Remi Langevin

Abstract. We construct an example of transitive Anosov flow on a compact 3-manifold, which admits a transversal torus and is not the suspension of an Anosov diffeomorphism.

0 Introduction

The simplest example of an Anosov flow on a manifold of dimension 3 is the suspension of the Thom-Anosov diffeormorphism of a matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ of $\mathbb{T}^2$. The fiber $\mathbb{T}^2$ is a torus transverse to this flow. The geodesic flow on the unit tangent bundle of a closed surface of Gaussian curvature -1 is also an Anosov flow which does not admit a transverse immersed torus.

These two flows are transitive, i.e., they admit a dense orbit.

In dimension greater than or equal to 4, Verjovsky [Ver74] shows that Anosov flows whose invariant foliation is of codimension 1 are transitive. This is false in dimension 3, since Franks and Williams [FW80] have constructed in 1980 the first examples of non-transitive Anosov flows.

According to Marco Brunella [Bru93] such non-transitive flows admit a transverse torus.

In January 1992 Etienne Ghys and Thierry Barbot posed the following question, (see [Bar92] remark 5.3.6):

Question. Let M be a compact manifold of dimension 3 and X a transitive Anosov flow transverse to a torus in M. Is the flow X topologically equivalent to a suspension?

We will show that the answer to this question is negative[1].

Theorem 0.1. *There exists a connected compact orientable manifold M, a field of vectors X on M which defines a transitive Anosov flow with the following properties: — there exists a torus $\mathbb{T}$ immersed in M transverse to X, — there exists a periodic orbit of X disjoint from $\mathbb{T}$.*

Corollary 0.2. *This field X is not conjugate to a suspension.*

[1] J. Christy tells us that from the examples of his article [Chr93] we could also build examples similar to ours.

Proof of the corollary. It suffices to show that, for any field of vectors X defined as the suspension of an Anosov diffemorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$, on a manifold V_f, and for any torus $\mathbb{T} \subset V_f$ transverse to X, any periodic orbit of X meets $\mathbb{T}$.

Indeed, V_f is torus bundle over $\mathbb{S}^1$, and, since f is of Anosov type, we have that $H_1(V_f, \mathbb{R}) = \mathbb{R}$, and that $H_1(V_f, \mathbb{R})$ is generated by the class of a section of the bundle. Any periodic orbit is homologous to a non-zero multiple of this generator.

The periodic orbits being dense in V_f, one of them, C, intersects $\mathbb{T}$. As the intersections are always made in the same orientation, the homological intersection of C with the torus $\mathbb{T}$ is non-zero. The same goes for any other periodic orbit, since its homology class is a non-zero rational multiple of C. ∎

Figure 1.

Let us give a first quick presentation of our example. Consider the field Y on $\mathbb{R}^2 \times \mathbb{S}^1$ defined by:

$$Y = f(x^2 + y^2)\partial/\partial\theta + x\partial/\partial x - y\partial/\partial y$$

where $f : \mathbb{R}^+ \to [0,1]$ is a half-hump function of support contained in $[0, 1/10]$. The circle $\gamma = 0 \times \mathbb{S}^1$ is a closed hyperbolic orbit of Y. Consider in $\mathbb{R}^2$ the domain:

$$\Delta = \{|x| \leq 1, |y| \leq 1; |xy| \leq 1\}$$

$\partial\Delta \times \mathbb{S}^1$ is a torus with corners which is a union of eight annuli, four tangent to Y and four transverse to Y.

By gluing together the annuli tangent to Y and the opposite in a manner compatible with the field Y we obtain a manifold of dimension 3 with the boundary of two tori $\mathbb{T}_1$ and $\mathbb{T}_2$ transverse to a field that we will still call Y. On these tori there are marked meridians, coming from the curves $\theta = 0$ of $\partial\Delta$ and the parallels coming from the curves $\{z\} \times \mathbb{S}^1, z \in \partial\Delta$. In gluing $\mathbb{T}_1$ and $\mathbb{T}_2$ by a map isotopic to the rotation of $\pi/2$ (modifying the meridian and the parallel, so that the stable and unstable manifolds $W^s(\gamma)$ and $W^u(\gamma)$ are transverse), we can obtain the example we seek for.

Examples of anosov flows in dimension 3 have already been constructed by analogous techniques by Goodman [Goo83] and Handel and Thurton [HT80], and more recently by Brunella [Bru94].

For technical reasons we will give here another construction. Before giving it, let us study the return map P on a transverse torus $\mathbb{T}$ of a hyperbolic flow X.

If P is defined everywhere then the flow is a suspension. In general, the map P is only defined from the domain $\mathrm{Dom}(P)$ to its image $\mathrm{Im}(P)$. The stable and unstable foliations of X have as traces on $\mathbb{T}$ the foliations f^s and f^u.

Lemma 0.3. *Dom(P) is saturated by f^s and Im(P) is saturated by f^u.*

Proof. If a curve leaves $\mathrm{Dom}(P)$, the return time of X starting from a point on this curve tends to infinity, on the other hand the derivative of this return time along a leaf of f^s is finite. ∎

This lemma restricts the possible choices of f^s, f^u and P. The simplest example consists of taking for f^s a foliation having two meridians, say $\mathbb{S}^1 \times 0$ and $\mathbb{S}^1 \times 1/2$, of $\mathbb{T} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ as compact leaves, while the other leaves spiral towards these meridians (without forming a Reeb component). The foliation f^u is constructed in a similar manner by having two parallels as compact leaves, see Figure 2.

Figure 2.

The domain of P is: $\mathbb{T} - \{$the compact leaves of $f^s\}$. The map P, on each annulus of $\mathrm{Dom}(P)$, sends the leaves of f^s to the parallel segments $a \times (0, 1/2)$ or $a \times (1/2, 0)$. See Figure 3.

Our example is essentially a 'suspension' of such a return map.

1 Construction of a manifold M_0 with boundary of two tori equipped with a transverse field on the boundary

In all of the following $\mathbb{S}^1$ will denote the circle $\mathbb{R}/4\mathbb{Z}$.

Denote by $\bar{N}$ the manifold with boundary $\bar{N} = \mathbb{R} \times [-1, 1] \setminus \bigcup_{i \in \mathbb{Z}} \mathbb{D}((2i, 0), 1/4)$, which we give the coordinates x, y, and $\bar{M}$ the product $\bar{M} = \bar{N} \times \mathbb{S}^1$, where we denote θ the last coordinate.

Let us denote by $\phi : \mathbb{R} \to [-1, 1]$ a function of class C^∞ satisfying:

Figure 3.

(1) ϕ is antisymmetric ($\phi(-x) = -\phi(x)$).
(2) For each $x \in \mathbb{R}$ we have: $\phi(x + 1) = -\phi(x)$;
(3) $\phi^{-1}(0) = \mathbb{Z}$;
(4) ϕ coincides with the identity on $[-1/3, 1/3]$.

Figure 4.

Let us denote by $\psi : \mathbb{R} \to [0, 1]$ a function of class C^∞ equal to 1 over $[-1/3, 1/3]$ and to 0 over $\mathbb{R} \setminus (-1/2, 1/2)$. See figure 4.

Denote by $\bar{X}$ the vector field over N defined by $\bar{X} = -\phi(x) \cdot \psi(x) \cdot \partial/\partial x - y \partial/\partial y$. It is a field on $\bar{N}$, transverse to the boundary, and whose time of passage from the boundary $\mathbb{R} \times \{-1, 1\}$ to the boundary $\bigcup_{i \in \mathbb{Z}} \partial\mathbb{D}((2i, 0), 1/4)$ tends to infinity when the point $(x, 1)$ or $(x, -1)$ tends to a point belonging to $(1 + 2\mathbb{Z}) \times \{-1, 1\}$. See Figure 5.

We will also denote by $\bar{X}$ the field of the same expression on $\bar{M}$.

Figure 5.

We denote by M_0 the compact 3-dimensional manifold with boundary equal to two tori $\mathbb{T}^2$, obtained by quotienting $\bar{M}$ by the diffeomorphism $\Phi : \bar{M} \to \bar{M}$ defined by:

$$\Phi(x, y, \theta) = (x + 2, -y, -\theta).$$

We will denote by $\mathbb{T}_1$ the torus corresponding to $|y| = 1$ and $\mathbb{T}_2$ the torus corresponding to $x^2 + y^2 = 1/16$. These tori are the two connected components of the boundary of M_0.

The field $\bar{X}$ constructed on $\bar{M}$ is invariant under the diffeomorphism Φ, and therefore passes to the quotient to a vector field on M_0 which we will denote by X_0. The field X_0 is transverse to the two tori $\mathbb{T}_1$ and $\mathbb{T}_2$ and admits a curve of fixed points: $(1, 0) \times \mathbb{S}^1$.

Let us now add to X_0 a component on the fiber $\mathbb{S}^1$ in the neighborhood of this curve of fixed points, and in the domain of M_0 (which is topologically of the form $\mathbb{T}^2 \times I$) quotient of $\mathbb{R} \times \{[-2/3, -1/2] \cup [1/2, 2/3]\} \times \mathbb{S}^1$.

Let us denote by $\alpha : \mathbb{R}^2 \to \mathbb{R}$ a function of class C^∞ having the following properties:

(1) $\forall x \in \mathbb{R}^2$, $\alpha(x + 2, y) = -\alpha(x, y)$;
(2) α is identically zero outside the disks $\mathbb{D}((2i + 1, 0), 1/3), i \in \mathbb{Z}$
(3) On the disk $\mathbb{D}((1, 0), 1/3)$, α has values in $[0, 1]$, depending only on the radius $((x - 1)^2 + y^2)^{1/2}$, is equal to 1 for very small radii and to 0 for radii close to 1/3.

Let us denote by $\beta : [0, 1] \to [0, 1]$ a function of class C^∞ vanising outside $(1/2, 2/3)$ and strictly positive over this interval.

Let: $\bar{Y}_t = -(\alpha(x, y) + t \cdot \beta(|y|) \cdot \sin((\Pi/2) \cdot x))\partial/\partial\theta$. This field is passed to the quotient a field on M_0 that we will denote by Y_t. We finally obtain a field Z_t on M_0 by setting $Z_t = X_0 + Y_t$.

2 Holonomy of the field Z_t, of $\mathbb{T}_1$ onto $\mathbb{T}_2$

The field $Z_t, t > 0$ is everywhere non-zero. It only has a periodic orbit, γ, which corresponds to the circle $(1, 0) \times \mathbb{S}^1$. This periodic orbit γ is disjoint from the boundary of M_0, and is hyperbolic of the saddle type. The planes $x = 2i + 1$ and

$y = 0$ induce on M_0 two cylinders invariant under the fields X_0 and Z_t. These cylinders are the respectively stable and unstable manifolds of γ.

The cylinder $x = 2i + 1$ induces on $\mathbb{T}_1$ two circles which cut $\mathbb{T}_1$ into two annuli. Likewise, $y = 0$ cuts $\mathbb{T}_2$ into two annuli.

Figure 6.

Any orbit of Z_t not contained in the invariant manifolds of the periodic orbit γ, enters it from the boundary $\mathbb{T}_1$ and leaves from $\mathbb{T}_2$. The map which, at the entry point associates the exit point induces a diffeomorphism from $\mathbb{T}_1 \backslash W^s(\gamma)$ to $T_2 \backslash W^s(\gamma)$, is called the holonomy of Z_t, of $\mathbb{T}_1$ onto $\mathbb{T}_2$, and denoted as h_t.

To describe the holonomy h_t, we will need coordinates on the tori $\mathbb{T}_1$ and $\mathbb{T}_2$.

$\mathbb{T}_1$ is given the system of coordinates (x, θ), $x \in \mathbb{R}/4\mathbb{Z}$, $\theta \in \mathbb{S}^1$, naturally induces coordinates on $\bar{M}$: the point (x, θ) of $\mathbb{T}_1$ corresponds to the point $(x, 1, \theta)$ of $\bar{M}$. $\mathbb{T}_1 \backslash W^s(\gamma)$ is the union of the annuli:

$$C_1^+ = \{(x, \theta) \mid x \in (-1, 1)\}, \quad C_1^- = \{(x, \theta) \mid x \in (1, 3)\}.$$

Denote by $U = \bar{M} \cap \{x \in (-1, 1)\}$. The closure $\bar{U}$ of U is a fundamental domain of the diffeomorphism. In this chart, C_1^+ is identified with the annulus $U \cap \{y = 1\}$ by $(x, \theta) = (x, 1, \theta)$, and C_1^- is identified with the annulus $U \cap \{y = -1\}$ by $(x, \theta) = (x - 2, -1, -\theta)$.

In the chart U, the torus $\mathbb{T}_2$ is the product $\Sigma \times \mathbb{S}^1$ where Σ is the circle $x^2 + y^2 = 1/16 \subset \mathbb{R}^2$.

Lemma 2.1. *There exists a function $\omega : \bar{N} \to \mathbb{R}/4\mathbb{Z}$ which has for level sets the orbits of $\bar{X}$ such that $\omega(x, 1) = x \mod 4$.*

Proof. The function ω is defined from the orbits of the field $\bar{X}$. The only thing to verify is the differentiability of ω at the points of the axis $y = 0$ and the fact that at points of the axis $y = 0$ different from $(0, \pm 1)$, $\partial\omega/\partial y \neq 0$. This differentiability is due to the fact that $\bar{X}$ admits a first integral in the neighborhood of the points $(0, \pm 1) \in \bar{N}$. ■

Lemma 2.1 allows us to choose ω for the coordinate also on the circle Σ.

Figure 7.

Let us now see how to calculate the holonomy h_t of Z_t: the orbits of Z_t differ from those of $\bar{X}$ by their component in $\partial/\partial\theta$ (in particular they have the same projections on the plane $\mathbb{R}^2$). To obtain h_t we must therefore essentially add to $h_{\bar{X}}$ the deviation in the direction θ of the orbits of Z_t. This deviation has two terms, one coming from the passage of the orbits in the support of α, and the other from the passage of the orbits in the support of β.

Crossing the support of β, the orbits are deviated by $-t\cdot\sin((\Pi/2)\cdot x)\cdot k$, where $k = \int_{-\infty}^{+\infty}\beta(e^t)\,dt$, since the x coordinate is constant along the orbits in the domain where β is different from 0.

Denote by $f:(-1,1)\to\mathbb{R}$ the deviation in the direction θ of the orbit passing through $(x,1,0)$ when it crosses the support of α. We verify that f tends to $+\infty$ at -1, to $-\infty$ at 1 and that its derivative is always negative or zero, and tends to $-\infty$ at 1 and at -1.

The holonomy h_t is then defined by:

$$h_t(x,1,\theta) = (x, \theta + f(x) - t\cdot\sin((\Pi/2)x)\cdot k)$$

$$h_t(x,-1,\theta) = (2-x, \theta + f(x) - t\cdot\sin((\Pi/2)x)\cdot k)$$

that is to say in the chosen coordinates (x,θ) on $\mathbb{T}_1$ and (ω,θ) on $\mathbb{T}_2 = \Sigma\times\mathbb{S}^1$.

$$\forall x\in(-1,1),\ h_t(x,\theta) = (x,\theta + f(x) - t\cdot\sin((\Pi/2)x)\cdot k)$$

and, remembering that $\alpha(x+2,y) = -\alpha(x,y)$,

$$\forall x\in(1,3),\ h_t(x,\theta) = (4-x,\theta + f(x-2) + t\cdot\sin((\Pi/2)x)\cdot k).$$

3 A vector field of Anosov type

Denote by $A:\mathbb{T}_1\to\mathbb{T}_2$ the map defined by $A(x,\theta) = (\theta,-x)$, and denote by M the manifold obtained by identifying $\mathbb{T}_2$ with $\mathbb{T}_1$ by A, and denote by $X_t, t>0$, the field induced by Z_t; there exists a differentiable structure on M compatible with that of M_0 and making X_t differentiable.

Lemma 3.1. *The field X_t preserves a volume form.*

Proof. Let us first show that the field Z_t preserves a volume. Note that the field $\bar{X}$ preserves on $\bar{N}\backslash(1,0)\cdot(2\mathbb{Z}+1)$ the form $d\omega\wedge d\tau$, where the τ is the travel time on the orbits of the field $\bar{X}$. This form coincides on $(0,\pm1)\times\mathbb{R}$ with the form $dx\wedge dy$.

In the neighborhood of points $(1,0)\cdot(2\mathbb{Z}+1)$ the form $dx\wedge dy$ is invariant under $\bar{X}$. Indeed, let ϑ be a neighborhood of $(1,0)$ on which $\bar{X} = (x-1)\partial/\partial x - y\partial/\partial y$, so on ϑ the form $dx\wedge dy$ is invariant under $\bar{X}$.

On $\vartheta\backslash(1,0)$ the two forms $d\omega\wedge d\tau$ and $dx\wedge dy$ are proportional. They are invariant under $\bar{X}$, so we have the equality: $d\omega\wedge d\tau = \delta(\omega)\cdot dx\wedge dy$ where the function $\delta(\omega)$ is continuous, positive, bounded and bounded away from zero, as we see by examining the equality on the boundary of ϑ . This shows that $d\omega\wedge d\tau$ extends at $(1,0)$ by $\delta(1)\cdot dx\wedge dy$, and thus extends in the same way at the singularities $(1,0)\cdot(2\mathbb{Z}+1)$ of $\bar{X}$. As the component of Z_t with $\partial/\partial\theta$ does not depend on θ the field Z_t preserves the volume $d\omega\wedge d\tau\wedge d\theta$. On the two boundary components the volume is, up to a sign, $d\omega\wedge d\theta\wedge d\tau$ and $dx\wedge d\theta\wedge d\tau$, respectively. As the image of $dx\wedge d\theta$ by the gluing function A is $d\omega\wedge d\theta$, the field X_t also preserves the volume $d\omega\wedge d\theta\wedge d\tau$. ∎

Remark. We deduce that the field X_t is transitive.

Denote by $\mathbb{T}$ the torus, transverse to X_t, corresponding to the tori $\mathbb{T}_i$ of M_0, and give $\mathbb{T}$ with the coordinates (x, θ) induced by those of $\mathbb{T}_1$.

The composition $A \circ h_t$ then induces on $\mathbb{T}$ the first return map on $\mathbb{T}$ of the field X_t, which we will denote by P_t. In the coordinates (x, θ), P_t is written:

$$\forall x \in (-1,1),\ P_t(x,\theta) = (\theta + f(x) - t \cdot \sin((\Pi/2)x) \cdot k, -x)$$

$$\forall x \in (1,3),\ P_t(x,\theta) = (-\theta + f(x-2) + t \cdot \sin((\Pi/2)x) \cdot k, x-4).$$

Proof of the theorem. Let us show that for $t > 0$ large enough, X_t is an Anosov flow satisfying the conditions of the theorem. Let us fix on the torus $\mathbb{T}$ a Riemannian metric which makes $(\partial/\partial x, \partial/\partial\theta)$ an orthonormal basis.

Lemma 3.2. *For all $t > 0$ large enough, there exists on the torus $\mathbb{T}$ a continuous cone field, $\{c^u(p)\}_{p \in \mathbb{T}}$ with the following properties:*

(1) For all $p \in \mathbb{T}$, $c^u(p)$ is a symmetric closed cone in the tangent space at the point p in $\mathbb{T}$. Moreover $c^u(p)$ varies continuously with p.
(2) For any p of the form $(x, \pm 1)$, the cone $c^u(p)$ is reduced to the line directed by $\partial/\partial x$.
(3) For any p where P_t is defined, i.e. the points $p = (x,\theta), x \neq \pm 1$, the image $D_pP_t(c^u(p))$ of the cone $c^u(p)$ by the differential of P_t at p is included in the interior of the cone $c^u(P_t(p))$.
(4) There is $\lambda > 1$ such as for all $p = (x,\theta), x \neq \pm 1$, for all vector $v \in c^u(p)$, we have:

$$\lambda \|v\| \leq \|D_pP_t(v)\|.$$

Proof. It is essentially necessary to calculate the differential of the function P_t. It is defined by:

$$\forall x \in (-1,1),\ D_{(x,\theta)}P_t(\partial/\partial\theta) = \partial/\partial x,$$

$$\forall x \in (1,3),\ D_{(x,\theta)}P_t(\partial/\partial\theta) = -\partial/\partial x,$$

$$\forall x \in (-1,1), D_{(x,\theta)}P_t(\partial/\partial x) = -\partial/\partial\theta + (f'(x) - tk(\Pi/2)\cos((\Pi/2)x))\partial/\partial x,$$

$$\forall \in (1,3),\ D_{(x,\theta)}P_t(\partial/\partial x) = +\partial/\partial\theta + (f'(x-2) - tk(\Pi/2)\cos((\Pi/2)(x-2)))\partial/\partial x.$$

What is important in these formulas is that the term $(f'(x) - tk(\Pi/2)\cos((\Pi/2)x))$ is strictly negative when $x \in (-1,1)$, tends to $-\infty$ when x approaches ± 1; moreover, its modulus can be reduced by an arbitrary constant, (which one chooses therefore very large) when one chooses $t > 0$ sufficiently large.

The same is true when $x \in (1,3)$, since the formula is the same, by replacing x by $(x-2) \in (-1,1)$.

Let us denote by $\tilde{c}$ the constant cone field defined by:

$$a\partial/\partial x + b\partial/\partial\theta \in \tilde{c} \Leftrightarrow |a| \geq 2|b|.$$

The cone field $DP_t(\tilde{c})$ is a priori defined on the image of P_t, i.e. on $\{\theta \neq \pm 1\}$. However $P_t(p)$ tends to $\{\theta = \pm 1\}$ if and only if p tends to $\{x = \pm 1\}$, and therefore if and only if f' tends to ∞, which implies that the cone $D_pP_t(\tilde{c})$ degenerates into the line led by $\partial/\partial x$.

Note that, for $t > 0$ large enough, the differential of the map P_t in the coordinate system $\partial/\partial x, \partial/\partial\theta$ is of the form: $\left[\begin{smallmatrix} -A & 1 \\ \pm 1 & 0 \end{smallmatrix}\right]$ where A is a function of x and of t greater than a large positive constant A_0. The two eigenvalues of the matrix, the roots of $\lambda^2 + A\lambda + 1 = 0$, are negative, and the modulus of one is of the order of A and the modulus of the other is of the order of $1/(A)$. The eigen-direction corresponding to the eigenvalue λ of modulus of the order of A is if $x \in (-1, 1)$ and therefore $\pm 1 = -1$:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \text{ is a solution of } \begin{bmatrix} -A & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

we must therefore have $-\alpha$ of the order of $\lambda\beta$, so $|\beta| \ll |\alpha|$, which implies that this eigen-direction is close to the x-axis. We also show that a directing vector $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$ of the eigen-direction corresponding to the eigenvalue of modulus of the order of $1/A$ satisfies $|\beta| \gg |\alpha|$ and therefore that the corresponding eigen-direction is close to the y-axis (the case where $x \in (1, 3)$, $\pm 1 = +1$ is analogous).

The position of the eigen-directions and the size of the eigenvalues imply that for any sufficiently large t, c_t^u is contained in the interior of $\tilde{c}$, and we can deduce that $DP_t(c_t^u)$ is a cone field defined on $\{\theta \neq \pm 1\}$, contained in the interior of c_t^u, and therefore extending by continuity over $\{\theta = \pm 1\}$. ■

Lemma 3.3 (Lemma 3.2 bis)**.** *For any $t > 0$ large enough, there exists on the torus $\mathbb{T}$ a continuous cone field, $\{c^s(p)\}_{p \in \mathbb{T}}$, with the following properties:*

(1) For all $p \in \mathbb{T}$, $c^s(p)$ is a symmetric closed cone in the tangent space at the point $p \in \mathbb{T}$, $c^s(p)$ varies continuously with p, and the intersection $c^u(p) \cap c^s(p)$ is reduced to the origin of the tangent space at p to $\mathbb{T}$.

(2) For any p of the form $(\pm 1, \theta)$, the cone $c^s(p)$ is reduced to the line directed by $\partial/\partial\theta$.

(3) For any p where P_t^{-1} is defined, i.e. the points $p = (x, \theta)$, $\theta \neq \pm 1$, the image $D_pP_t^{-1}(c^s(p))$ of cone $c^s(p)$ by the differential of P_t^{-1} at p is included in the interior of the cone $c^s(P_t^{-1}(p))$.

(4) There exists $\lambda > 1$ such that for all $p = (x, \theta)$, $\theta \neq \pm 1$, for any vector $v \in c^s(p)$, we have:

$$\lambda \|v\| \leq \|D_pP_t^{-1}(v)\|.$$

Proof. The matrix of $D_pP_t^{-1}$ is of the form $\left[\begin{smallmatrix} 0 & 1 \\ -1 & -A \end{smallmatrix}\right]$ (if $x \in (-1, 1)$), the 2 eigenvalues are roots of $\lambda^2 + A\lambda + 1 = 0$ and as previously the modulus of one is of the order of A, and that of the other is of the order of $1/A$. The eigen-directions of which a directing vector is of the form:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \text{ is a solution of } \begin{bmatrix} 0 & 1 \\ -1 & -A \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

are again close to the axes, which makes it possible to repeat the previous reasoning. the case $x \in (1,3)$ is analogous. ∎

The existence of the cone fields c^u and c^s leads to:

Corollary 3.4. *For $t > 0$ large enough, there exists on the torus $\mathbb{T}$ two line fields l^s and l^u, invariant under P_t, and satisfying:*

(1) For all $p \in (x = \pm 1)$, $l^s(p)$ is the line directed by $\partial/\partial\theta$;
(2) For all $p \in (\theta = \pm 1)$, $l^s(p)$ is the line directed by $\partial/\partial x$;
(3) For any p where P_t is defined, for any $v \in l^u(p)$ we have:

$$\|v\| \leq \lambda \|D_p P_t(v)\|, \quad \textit{where} \quad \lambda > 1;$$

(4) For all p where P_t^{-1} is defined, for all $v \in l^s(p)$ we have:

$$\lambda \|v\| \leq \|D_p P_t^{-1}(v)\|, \quad \textit{where} \quad \lambda > 1.$$

Proof. We show that $\bigcap_{n=0}^{n=\infty} DP_t^n(c^u)$ is a decreasing intersection of cone fields which converges to a continuous line field l^u, which verifies, of course, the points (2) and (3) of the corollary. Likewise, $\bigcap_{n=0}^{n=\infty} DP_t^{-n}(c^s)$ is a continuous line field l^s which verifies the points (1) and (4) of the corollary. ∎

End of the proof of the theorem. From now on we will denote by X the field X_t, for a fixed value of t large enough to be able to apply Lemmas 3.2 and 3.3 bis and Corollary 3.4. We will denote by $P = P_t$ its first return map on $\mathbb{T}$.

Recall that γ is a periodic orbit of the field X, and that it is the only orbit of X not meeting the transverse torus $\mathbb{T}$. Let us denote by τ^u the 2-plane field defined on $M \backslash \gamma$ as follows:

(1) For all $p \in \mathbb{T}$, $\tau^u(p)$ is the plane generated by $X(p)$ and by the line $l^u(p)$.
(2) For all $p \in M \backslash \gamma$ the orbit of p meets $\mathbb{T}$ in least one point q. The plane $\tau^u(p)$ is then the image of $\tau^u(q)$ under the differential of the flow of X. The fact that l^u is invariant under P allows us to show that $\tau^u(p)$ is well-defined in a unique way of this manner.

We define in an analogous way the field of 2-planes τ^s on $M \backslash \gamma$, invariant under the flow of X and defined at any point of $\mathbb{T}$ by X and the line l^s.

The following lemma completes the proof of the theorem:

Lemma 3.5. *(1) The plane fields τ^u and τ^s are transverse on $M \backslash \gamma$.*

(2) The plane fields τ^u and τ^s are extended by continuity on γ to two fields of transverse planes, and tangent respectively to $W^u(\gamma)$ and $W^s(\gamma)$.

Proof. Item (1) is obtained by noting that, since τ^u and τ^s are transverse on $\mathbb{T}$, they remain so when we transport them in $M \backslash \gamma$ by the flow of X. Item (2) is essentially a consequence of the 'λ-lemma': τ^u is a plane field containing the field of vectors X, invariant under the flow of X and transverse to $W^s(\gamma)$. The

'λ-lemma' ensures that it extends by continuity into a field of planes tangent to $W^u(\gamma)$. Same thing for τ^s. ■

Remark. The manifold M is a graph manifold, i.e., if we cut it along a finite number of disjoint immersed tori, we get a circle bundle over a compact surface as the boundary (see Waldhausen [Wal67]). In fact the manifold M_0 of §2 is a circle bundle with the base a projective plane taken off two discs, and M is obtained by gluing the two boundary components of M_0.

References

[Bar92] Thierry Barbot. *Géométrie transverse des flots d'Anosov*. PhD thesis, Lyon 1, 1992.

[Bru93] Marco Brunella. Separating the basic sets of a nontransitive anosov flow. *Bulletin of the London Mathematical Society*, 25(5):487–490, 1993.

[Bru94] Marco Brunella. On the discrete godbillon-vey invariant and dehn surgery on geodesic flows. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, volume 3, pages 335–344, 1994.

[Chr93] Joe Christy. Branched surfaces and attractors. i: Dynamic branched surfaces. *Transactions of the American Mathematical Society*, pages 759–784, 1993.

[FW80] John Franks and Bob Williams. Anomalous anosov flows. In Zbigniew Nitecki and Clark Robinson, editors, *Global Theory of Dynamical Systems*, pages 158–174, Berlin, Heidelberg, 1980. Springer Berlin Heidelberg.

[Goo83] Sue Goodman. Dehn surgery on anosov flows. In J. Palis, editor, *Geometric Dynamics*, pages 300–307, Berlin, Heidelberg, 1983. Springer Berlin Heidelberg.

[HT80] Michael Handel and William P Thurston. Anosov flows on new three manifolds. *Inventiones mathematicae*, 59(2):95–103, 1980.

[Ver74] Alberto Verjovsky. Codimension one anosov flows. *Bol. Soc. Mat. Mexicana*, 19(2):49–77, 1974.

[Wal67] Friedhelm Waldhausen. Eine klasse von 3-dimensionalen mannigfaltigkeiten. i. *Inventiones mathematicae*, 3(4):308–333, 1967.

Translated by Danyu Zhang (zhang.8939@osu.edu), October 2021