ANOSOV DIFFEOMORPHISMS ON A PRODUCT OF SURFACES

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Abstract. We show that there is no transitive Anosov diffeomorphism with the global product structure, on a product of two closed surfaces, each of which has genus greater than or equal to two.

1 Introduction

We start by recalling the definition of an Anosov diffeomorphism and some progress on the classification problems of Anosov diffeomorphisms. Let $f : M \to M$ be a C^1 diffeomorphism. If there exist an invariant splitting $TM = E^s \oplus E^u$ under df, a Riemannian metric $\|\cdot\|$ on M, and constants $C > 0, \lambda \in (0,1)$, such that

$$\|df^n v^s\| \leq C\lambda^n \|v^s\|, \quad \text{for } v^s \in E^s,$$

$$\|df^{-n} v^u\| \leq C\lambda^n \|v^u\|, \quad \text{for } v^u \in E^u,$$

then we call f an Anosov diffeomorphism. A dynamical system is *transitive* if there exists a dense orbit.

It is a long-time open question (see for example [Sma98]) if every Anosov diffeomorphism lives on up to a finite cover a nilmanifold (a manifold whose universal cover is a simply connected nilpotent Lie group). Codimension-one (i.e. either the stable or unstable bundle has dimension 1) Anosov diffeomorphisms only live on torus, according to Franks [Fra70] and Newhouse [New70]. Yano [Yan83] has showed that there are no transitive Anosov diffeomorphisms on negatively curved manifolds. Gogolev and Lafont [GL16] proved that a product $M_1 \times \cdots \times M_k \times N$ where $M_i,\ i = 1, \dots, k$ is a closed negatively curved manifold of dimension ≥ 3 and N is a nilmanifold does not admit transitive Anosov diffeomorphisms. Neofytidis [Neo21] showed that any geometrizable 4-manifold which is not finitely covered by a product of closed surfaces $S_g \times S_h$ of genus g, h where $g \geq 2$ *or* $h \geq 2$ does not admit transitive Anosov diffeomorphisms.

Consider the pair of the stable and unstable foliations of an Anosov diffeomorphism lifted to the universal cover, if every pair of the stable and unstable leaves in the universal cover intersects only once, then we say that the Anosov diffeomorphism has the *global product structure*. Hammerlindl [Ham14] showed that if an Anosov diffeomorphism has the *polynomial global product structure*, (i.e., suppose x, y denote two points in the universal cover, z denotes the unique intersection of the stable leaf passing through x and the unstable leaf passing through y, and d_s, d_u denote the distance along the stable and unstable leaves respectively, while d is the extrinsic metric, and then there exists a polynomial p such that $d_s(x,z) + d_u(y,z) < p(d(x,y))$), then it is topologically conjugate to an infranilmanifold automorphism. The polynomial global product structure implies that the fundamental group has polynomial growth.

In this short note, we want to partially answer the question asked in [GL16], whether there is an Anosov diffeomorphism on a product of two hyperbolic surfaces. We want to point out that this case is interesting to us because it lies in between the above already known cases: The 4-manifold is a product of two hyperbolic manifolds of dimension 2, and the leaves do not have the polynomial global product structure.

Theorem 1.1. *There is no transitive Anosov diffeomorphism with the global product structure on a product of two closed surfaces of genera each* ≥ 2.

We want to proceed by reaching a contradiction. Let S_1, S_2 denote the two surfaces, each of which has genus ≥ 2, and $M := S_1 \times S_2$. Suppose $G : M \to M$ is a transitive Anosov diffeomorphism with the global product structure.

We follow the following steps.

1. We show that up to homotopy, G can be reduced to a product of self-diffeomorphisms f_1, f_2 of each surfaces such that

$$G_{\#} = (f_1)_{\#} \times (f_2)_{\#} : \pi_1(S_1) \times \pi_1(S_2) \to \pi_1(S_1) \times \pi_1(S_2),$$

and G is homotopic to $f_1 \times f_2 =: F$.
2. We show that both f_1 and f_2 have to be pseudo-Anosov.
3. We establish Handel's result [Han85] for $f_1 \times f_2$, i.e., we show that there exists a closed G-invaraint subset Y and a continuous surjective map $\varphi : Y \to M$, such that $\varphi \circ G|_Y = F \circ \varphi$.
3. We show that φ is a homeomorphism by adjusting the argument of Handel [Han85] for the Anosov diffeomorphism. Hence there is a contradiction.

Before we commence to give the details of the proof, we would also like to point out two apparent questions about Anosov diffeomorphisms that arise constantly: 1. Are all Anosov diffeomorphisms transitive? 2. Do all Anosov diffeomorphisms have the global product structure?

2 The proof of 1.1

2.1 Preliminaries and notations

We further recall several common notations and facts that we will use in our proof.

The stable and unstable distributions of an Anosov diffeomorphism on a closed manifold M are integrable to what we call the stable and unstable leaves. We denote the local embeded discs of diameter ε of the stable and unstable leaves centered at a point $x \in M$ as $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ respectively.

The local leaves also always have the *local product structure*, that is, there exist constants $\varepsilon > 0$, $\delta > 0$ such that for all $x, y \in M$ with $d(x,y) < \delta$, $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ (also $W^u_\varepsilon(x)$ and $W^s_\varepsilon(y)$) intersect transversely in a unique point. We remark that ε, δ do not depend on x, y.

Here is an important property that follows from the global and local product structure, which we make use of repeatedly in later proofs.

Remark 2.1. Suppose that a pair of foliations $\mathscr{F}_1, \mathscr{F}_2$ in $\tilde{M}$ has both the global and the local product structure. Then for any compact set $K \subseteq \tilde{M}$, there exists a constant B_K such that

$$\sup_{x \in K}\{d_i(x,y) : \ y \in K \text{ is in the same leaf of } \mathscr{F}_i \text{ with } x\} < B_K, \quad i = 1, 2,$$

where d_i denotes the distance along the leaf between two points in the same leaf of $\mathscr{F}_i$. This seems to be implicit in the proof of Theorem 1 of [Fra69], and we gave a detailed fact checking in Lemma 5.6 of [Zha24].

We also want to take the following two definitions from [Han85]. Suppose we have $F : M \to M$ and $G : M \to M$ such that $F \simeq G$.

Definition 2.2. *The F-orbit of $x \in M$ is K-*globally shadowed *by the G-orbit of $y \in M$ if there are lifts $\tilde{x}$ of x and $\tilde{y}$ of y such that the distance in the universal cover between corresponding points $d(\tilde{F}^k\tilde{x}, \tilde{G}^k\tilde{y}) \leq K$ for all $k \in \mathbb{Z}$. We write $(F,x) \sim^K (G,y)$ or $(F,x) \sim (G,y)$ if the shadowing constant K is not specified.*

Definition 2.3. *If $x \in M$ is a fixed point of F^n and $\tilde{x}$ is a lift of x, then $\tilde{F}^n\tilde{x} = s\tilde{x}$ for some covering translation s of $\tilde{M}$. Similarly, if $y \in M$ is a fixed point of G^n and $\tilde{y}$ is a lift of y, then $\tilde{G}^n\tilde{y} = t\tilde{y}$ for some covering translation t. We say that (F^n, x) and (G^n, y) are* Nielsen equivalent *if there exist lifts $\tilde{x}$ and $\tilde{y}$ such that $s = t$.*

Remark 2.4. Both K-global shadowing and Nielsen equivalence define equivalence relations.

Remark 2.5. One useful observation that we want to make is that the definition of Nielsen equivalence above is equivalent to "H-related" [Bro71], i.e., if F is homotopic to G by a homotopy $H : M \times I \to M$, and x, y are fixed points of F and G respectively, there exists a path $C : I \to M$ such that $H(C(t), t)$ is homotopic to $C(t)$ relative to x, y.

Indeed, if there exist lifts $\tilde{x}, \tilde{y}$ such that $\tilde{F}^n\tilde{x} = s\tilde{x}$ and $\tilde{G}^n\tilde{y} = s\tilde{y}$ where $s \in \pi_1(M)$, then by path-connectedness of $\tilde{M}$, take any path C connecting $\tilde{x}, \tilde{y}$, $\tilde{H}(C(t), t)$ is homotopic to $sC(t)$ which is just another lift of the same path, because $\tilde{M}$ is simply-connected. Thus the projection of C to M is a path that we want.

Conversely, if there exists a path C in M such that $C(0) = F^n x = x$, $C(1) = G^n y = y$, and $H(C(t), t)$ is homotopic to C where H is a homotopy between F and G, then the lift of the homotopy between $H(C(t), t)$ and C tells us that $\tilde{F}^n\tilde{x} = s\tilde{C}(0)$ and $\tilde{F}^n\tilde{y} = s\tilde{C}(1)$ because the fibres of the covering projection are discrete.

Throughout this note, we use S_1, S_2 to denote the two closed surfaces of genera ≥ 2 of our interest, and $M = S_1 \times S_2$. We are always going to let $f_\#$ denote the induced automorphism of a map $f : N \to N$ on $\pi_1(N)$.

2.2 Reduction of the map

Let us denote $\Gamma_1 := \pi_1(S_1)$, $\Gamma_2 := \pi_1(S_2)$, and $\Gamma := \Gamma_1 \times \Gamma_2$.

Lemma 2.6. *Let $\psi \in Aut(\Gamma)$. Then $\psi^2 = \psi_1 \times \psi_2$, where $\psi_i \in Aut(\Gamma_i)$, $i = 1, 2$.*

Proof. Let $(g, h) \in \Gamma$. The centralizer $C_\Gamma((g, h)) = C_{\Gamma_1}(g) \times C_{\Gamma_2}(h)$. In a surface group G (of a surface of genus ≥ 2), if $G \ni x \neq \text{id}$, then $C_G(x) = \mathbb{Z}$ ([FM11], pg. 23); if $x = \text{id}$, $C_G(x) = G$.

For any $\psi \in \text{Aut}(\Gamma)$, we have $C_\Gamma(x) \xrightarrow[\psi]{\sim} C_\Gamma(\psi(x))$.

Now if $g = \text{id}$ but $h \neq \text{id}$, $C_\Gamma((g, h)) \simeq \Gamma_1 \times \mathbb{Z} \simeq C_\Gamma(\psi(g, h))$. So $\psi(g, h) = (\text{id}, h')$ where $h' \neq \text{id}$. This means $\psi(\langle\text{id}\rangle \times \Gamma_2) = \langle\text{id}\rangle \times \Gamma_2$ or $\Gamma_1 \times \langle\text{id}\rangle$ if $\Gamma_1 \simeq \Gamma_2$. Similarly $\psi(\Gamma_1 \times \langle\text{id}\rangle) = \Gamma_1 \times \langle\text{id}\rangle$ or $\langle\text{id}\rangle \times \Gamma_2$. ∎

Again let $G : M \to M$ where $M = S_1 \times S_2$. Without loss of generality we assume that $G_\#$ splits on $\pi_1(M)$. Otherwise we take G^2. In the rest of the paper, we pick and fix $f_1 : S_1 \to S_1$ and $f_2 : S_2 \to S_2$, surface homeomorphisms that induce the automorphsms on the fundamental groups, as in the above lemma, such that $G \simeq f_1 \times f_2$.

2.3 Handel for a product of surfaces

Now suppose $G : M \to M$ is an Anosov diffeomorphism and it is homotopic to $f_1 \times f_2$ where $f_i : S_i \to S_i$, $i = 1, 2$ are surface diffeomorphisms. We let p_1, p_2 denote the projections $p_1 : M \to S_1$ and $p_2 : M \to S_2$.

We are always going to assume that our stable or unstable bundle of G has dimension 2. Otherwise it is covered by Franks-Newhouse [Fra70] [New70].

We then have the following observation.

Lemma 2.7. *While G is Anosov with the global product structure, f_1 and f_2 are both pseudo-Anosov.*

Proof. Without loss of generality, we assume that the mapping class of f_1 is reducible. We will reach a contradiction.

Consider an isotopy class of simple closed curves of S_1, such that the class is invariant under f_1. We take a representative simple closed curve γ of this isotopy class. We cut open γ at $x \in \gamma$. To suppress the notation, we also use γ and x to denote the curve and the point in a fixed copy of S_1 in the product space $S_1 \times S_2$, say $p_2^{-1}(y)$ where $y \in S_2$.

Now we could choose lifts of γ and x, and denote them as $\tilde{\gamma}$ and $\tilde{x}$. Also fix lifts $\tilde{f}_1, \tilde{f}_2$ and $\tilde{G}$. As $\tilde{f}_1 \times \tilde{f}_2 \simeq \tilde{G}$, and f_1 fixes the homotopy class of γ, we have that $\tilde{\gamma}$, $(\tilde{f}_1 \times \tilde{f}_2)^n(\tilde{\gamma})$ and $\tilde{G}^n(\tilde{\gamma})$ are freely homotopic for any $n > 0$. Let us denote $\tilde{G}^n(\tilde{x}) =: \tilde{x}_n$ and so the other endpoint of $\tilde{G}^n(\tilde{\gamma})$ is $\gamma\tilde{x}_n$, as it is $\tilde{x}$ translated by γ, thought as a member of $\pi_1(M)$.

Consider the projection along the stable leaf passing through $\gamma\tilde{x}_n$ and to the unstable leaf through the unique intersection point $\tilde{z}_n$ (see Figure 1). Note that with our definition of notation, $\tilde{G}(\tilde{x}_n) = \tilde{x}_{n+1}$, $\tilde{G}(\tilde{z}_n) = \tilde{z}_{n+1}$, as $\tilde{G}$ maps stable (unstable) leaves to stable leaves (unstable) leaves. The piece between $\gamma\tilde{x}_n$ and $\tilde{z}_n$ becomes shorter and the length of the piece between the points $\tilde{x}_n$ and $\tilde{z}_n$ grows exponentially as n grows.

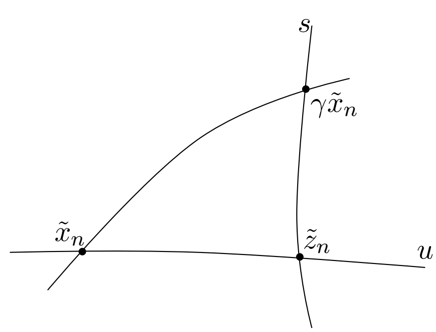

FIGURE 1. Curves under iteration; s and u mark the stable and unstable leaves respectively

But on the other hand, by the property of the global product structure 2.1, for any compact set K in $\tilde{M}$, there is a large $N > 0$ such that $\tilde{x}_n$ and $\tilde{z}_n$ grow out of αK, for any $\alpha \in \pi_1(M)$ (i.e. any translation of K), as long as the unstable leaf does not entirely lie in S_2, which is the case as the leaf has dimension 2. We can further take larger N to make sure that $\gamma\tilde{x}_n$ is so close to $\tilde{z}_n$ and whenever $n > N$, $\gamma\tilde{x}_n$ is also outside any translation of K.

Suppose now we take K to be a large set that contains any fundamental domain of $\tilde{M}$ and the translation of this fundamental domain under γ. Then we have reached a contradiction, as for large n, $G^n(\gamma)$ cannot be freely homotopic to γ. ∎

We have the following proposition.

Proposition 2.8. *Let f_1, f_2 be two pseudo-Anosov diffeomorphisms of two surfaces S_1, S_2, each of which has genus ≥ 2. Suppose $G : S_1 \times S_2 \to S_1 \times S_2$ is any map that is homotopic to $F := f_1 \times f_2$. Then there exists a closed subset $Y \subseteq S_1 \times S_2$ and a continuous surjective map $\varphi : Y \to S_1 \times S_2$, homotopic to the inclusion, such that $\varphi \circ G|_Y = F \circ \varphi$.*

We want to point out that the proof is directly generalized from [Han85]. We include the detailed proof for completeness in Appendix A.

2.4 The contradiction

In this subsection, we assume our Anosov diffeomorphsim G: 1. is transitive, which implies that periodic points are dense; 2. G has the global product structure.

Furthermore, from the last section, $G_\# : \pi_1(M) \to \pi_1(M)$ does not fix any free homotopy class neither.

Remark 2.9. We observe that a transitive Anosov diffeomorphism with the global product structure simulates the behavior of a pseudo-Anosov diffeomorphism, so we expect that we could run the argument of Handel one more time and get an inverse of the φ in Proposition 2.8.

Proposition 2.10. *The φ defined in Proposition 2.8 is a homeomorphism.*

To prove the proposition, we prove the Anosov version of following lemmas.

Lemma 2.11 (Lemma A.4 for Anosov)**.** *(i) If y_1, y_2 are distinct fixed points of G^n, then (G^n, y_1) and (G^n, y_2) are not Nielsen equivalent. (ii) If y is G-periodic with least period n, then there exists x which is F-periodic with least period n and such that (F^n, x) is Nielsen equivalent to (G^n, y).*

Proof. Suppose G^n fixes y_1 and y_2 and (G^n, y_1) and (G^n, y_2) are Nielsen equivalent, i.e., there exists lifts $\tilde{y}_1$, $\tilde{y}_2$ and $t \in \pi_1(M)$ such that $\tilde{G}^n\tilde{y}_1 = t\tilde{y}_1$ and $\tilde{G}^n\tilde{y}_2 = t\tilde{y}_2$. Then $t^{-1}\tilde{G}^n$, as another lift of $\tilde{G}^n$, fixes both $\tilde{y}_1$ and $\tilde{y}_2$. Suppose $\tilde{z}$ is the intersection of the unstable leaf of $\tilde{y}_1$ and stable leaf $\tilde{y}_2$. Then $\tilde{z}$ must also be fixed, which is impossible, because $t^{-1}\tilde{G}^n$ maps the unstable (stable) leaf to the unstable (stable) leaf and maps the intersection to the intersection.

Now the fixed point index of G can only be $(-1)^{\text{the dimension of the unstable bundle}} = \pm 1$. By Theorem 3, pg 94 of [Bro71], there exists a fixed point x of F^n such that (F^n, x) is Nielsen equivalent to (G^n, y). We can check that n is the least period of x , by the same argument as in the proof Lemma A.4. ∎

Corollary 2.12 (Theorem A.8 (ii) for Anosov)**.** *For all $y \in M$, there exists an $x \in M$ such that $(F, x) \sim (G, y)$; if y is G-periodic with least period n, then x can be chosen to be F-periodic with least period n.*

Proof. Lemmas A.2 and A.6 are still true because we are working with F. Note that since Nielsen equivalence and K-global shadowing are symmetric, because they are equivalence relations, for our fixed F, G, we can simply apply the statements for F.

Now because the periodic points of G are also dense, for any $y \in M$, there exists a sequence of periodic points y_n that approaches y and each of which is globally shadowed by a periodic point x_n of F, by Lemma 2.11. Then we can choose a convergent subsequence of x_n. The limit point globally shadows y, by Lemma A.6. It has least period n, which follows the same argument as in Lemma A.4 (see also Remark A.9). ∎

Proof of Proposition 2.10. Recall that in the proof of Proposition 2.8 (i.e. Theorem A.1 of the Appendix), we have defined

$$Y = \{y \in M : \text{ there exists } x \in M \text{ which globally shadows } y\},$$

and $\varphi : Y \to M$ such that $\varphi(y) = x$ where x globally shadows y. By Corollary 2.12, $Y = M$. We only need to show that φ is injective.

Assume that there are $y_1 \neq y_2$ such that $\varphi(y_1) = \varphi(y_2)$. Then there are lifts $\tilde{y}_1 \neq \tilde{y}_2$ and $\tilde{\varphi}$ such that $\tilde{\varphi}(\tilde{y}_1) = \tilde{\varphi}(\tilde{y}_2)$.

Consider the unique intersection $\tilde{z}$ of the unstable leaf u_1 of $\tilde{y}_1$ and the stable leaf s_2 of $\tilde{y}_2$. Let d_u and d_s denote the distances between two points along the unstable and stable leaves respectively. We know that

$$d_u(\tilde{G}^k\tilde{y}_1, \tilde{G}^k\tilde{z}) \geq C\mu^{-k}d_u(\tilde{y}_1,\tilde{z}), \quad \text{and} \tag{1}$$

$$d_s(\tilde{G}^{-k}\tilde{y}_2, \tilde{G}^{-k}\tilde{z}) \geq C\mu^{-k}d_s(\tilde{y}_2,\tilde{z}), \tag{2}$$

where $\mu \in (0,1)$ and $k > 0$.

From the topology of the measured foliations of the pseudo-Anosov maps, we know that each pair of stable and unstable leaves also intersect only once. Also, $\tilde{\varphi}$ maps local stable and unstable discs of $\tilde{G}$ to the local stable and unstable discs of $\tilde{F}$ respectively. Thus $\tilde{\varphi}(\tilde{z}) = \tilde{\varphi}(\tilde{y}_1) = \tilde{\varphi}(\tilde{y}_2)$. In addition, for any $n \in \mathbb{Z}$,

$$\tilde{\varphi}(\tilde{G}^n\tilde{y}_1) = \tilde{F}^n(\tilde{\varphi}\tilde{y}_1) = \tilde{F}^n(\tilde{\varphi}\tilde{y}_2) = \tilde{F}^n(\tilde{\varphi}\tilde{z}) = \tilde{\varphi}(\tilde{G}^n\tilde{y}_2) = \varphi(\tilde{G}^n\tilde{z}).$$

On the other hand, if we let K denote the preimage of a fundamental domain of M under $\tilde{\varphi}$. It is compact because $\tilde{\varphi}$ is proper. There is an upperbound $B_K > 0$ depending on K such that

$$\sup_{x\in K}\left\{d_u(x,y) : \ y \in u(x;\tilde{G})\right\} \leq B_K$$

where $u(x;\tilde{G})$ denotes the unstable leaf of $\tilde{G}$ passing through x (see Remark 2.1). But now (1) tells us that the distance between $\tilde{G}^n\tilde{y}_1$ and $\tilde{G}^n\tilde{z}$ along the leaf can be unbounded as $n \to \infty$. We have reached a contradiction. ■

Proof of the main theorem 1.1. The homeomorphism φ should map the leaves of G to the leaves of F, but this is impossible when it comes to a singular leaf of F. ■

Acknowledgements. I would like to thank Jean Lafont for pointing me to this question and many conversations.

A Proof of Handel for a product of surfaces

In this appendix, we let $f_1 : S_1 \to S_1$, $f_2 : S_2 \to S_2$ denote two pseudo-Anosov diffeomorphisms of closed surfaces S_1, S_2, denote $M := S_1 \times S_2$, and let $G : M \to M$ be any map that is homotopic to $F := f_1 \times f_2$.

We follow [Han85] to prove the following theorem.

Theorem A.1 (Theorem 2 of [Han85])**.** *There exists a closed G-invariant subset $Y \subseteq M$ and a continuous surjective map $\varphi : Y \to M$ that is homotopic to an inclusion, such that $\varphi \circ G|_Y = F \circ \varphi$.*

We will make use of the following properties for a pseudo-Anosov homeomorphism $f : S \to S$, where S is a closed surface.

(1) The periodic points of f are dense;
(2) The action induced by f on the free homotopy classes of S has no periodic orbits;
(3) The fixed point index of a fixed point x of f^n is never 0;
(4) There exist $\lambda > 1$ and an equivariant metric $\tilde{D}$ on the universal cover $\tilde{S}$ of S such that $\tilde{D} = \sqrt{\tilde{D}_s^2 + \tilde{D}_u^2}$, where $\tilde{D}_s : \tilde{S}\times\tilde{S} \to [0,\infty)$ and $\tilde{D}_u : \tilde{S}\times\tilde{S} \to [0,\infty)$ are equivariant functions satisfying:

$$\tilde{D}_u(\tilde{f}\tilde{x}_1, \tilde{f}\tilde{x}_2) = \lambda\tilde{D}_u(\tilde{x}_1,\tilde{x}_2) \quad \text{and} \quad \tilde{D}_s(\tilde{f}^{-1}\tilde{x}_1, \tilde{f}^{-1}\tilde{x}_2) = \lambda\tilde{D}_s(\tilde{x}_1,\tilde{x}_2)$$

for all $\tilde{x}_1, \tilde{x}_2 \in \tilde{S}$ and all lifts $\tilde{f}$ of f.

We now check the corresponding properties for F. Note (4') is different but will be sufficient for our purposes.

(1') The periodic points of F are dense.

Proof. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}$ be sets of periodic points of f_1, f_2 and F, respectively. Then if $(x_1, x_2) \in \mathcal{P}_1 \times \mathcal{P}_2$, there exist n_1, n_2 such that $f^{n_1} x_1 = x_1$ and $f^{n_2} x_2 = x_2$, so $F^N(x_1, x_2) = (x_1, x_2)$ for some N and $\mathcal{P}_1 \times \mathcal{P}_2 \subseteq \mathcal{P}$. Since the periodic points of f_1 and f_2 are dense, $\overline{\mathcal{P}}_1 = S_1$ and $\overline{\mathcal{P}}_2 = S_2$. For any $(x, y) \in M$, any neighborhood $U \times V$ that contains (x, y), there exist $x' \in U \cap \mathcal{P}_1, y' \in V \cap \mathcal{P}_2$, so $(x', y') \in U \times V \cap \mathcal{P}_1 \times \mathcal{P}_2 \subseteq \mathcal{P}$. Therefore $\overline{\mathcal{P}} \supset M$ and $\overline{\mathcal{P}} = M$. ∎

(2') The action induced by F on the free homotopy classes of M has no periodic orbits.

Proof. We know that $\pi_1(M) = \pi_1(S_1) \times \pi_1(S_2)$ and $F_\# = (f_1)_\# \times (f_2)_\#$. Suppose $F_\#$ has a periodic orbit. There exist $\alpha \in \pi_1(S_1)$, $\beta \in \pi_1(S_2)$, $\gamma \in \pi_1(M)$, and $n \in \mathbb{N}$ such that $\big((f_1)_\#^n \alpha, (f_2)_\#^n \beta\big) = \gamma^{-1}(\alpha, \beta)\gamma$. Then there exist periodic free homotopy classes of f_1, f_2, which contradicts to (2). ∎

(3') The fixed point index of a fixed point $(x, y) \in M$ of F^n is never 0.

Proof. By [Bro71], Theorem 6, p. 60., the index of a product is the product of the indices. ∎

(4') Let $\tilde{D}_1 = \sqrt{\tilde{D}_{1s}^2 + \tilde{D}_{1u}^2}, \tilde{D}_2 = \sqrt{\tilde{D}_{2s}^2 + \tilde{D}_{2u}^2}$ denote the equivariant metrics on the universal covers $\tilde{S}_1, \tilde{S}_2$ respectively, which satisfy (4) above.

For $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$, $\tilde{y} = (\tilde{y}_1, \tilde{y}_2) \in \tilde{M}$, define the product metric on $\tilde{M}$

$$\tilde{D}_s\big((\tilde{x}_1, \tilde{x}_2), (\tilde{y}_1, \tilde{y}_2)\big) = \sqrt{\tilde{D}_{1s}^2(\tilde{x}_1, \tilde{y}_1) + \tilde{D}_{2s}^2(\tilde{x}_2, \tilde{y}_2)},$$

$$\tilde{D}_u\big((\tilde{x}_1, \tilde{x}_2), (\tilde{y}_1, \tilde{y}_2)\big) = \sqrt{\tilde{D}_{1u}^2(\tilde{x}_1, \tilde{y}_1) + \tilde{D}_{2u}^2(\tilde{x}_2, \tilde{y}_2)},$$

and $\tilde{D} = \sqrt{\tilde{D}_s^2 + \tilde{D}_u^2} : \tilde{M} \times \tilde{M} \to [0, \infty)$. Then $\tilde{D}$ is an equivariant metric on $\tilde{M}$ because if $\alpha \in \pi_1(S_1)$ and $\beta \in \pi_1(S_2)$,

$$\begin{aligned}\tilde{D}\big((\alpha, \beta)(\tilde{x}_1, \tilde{x}_2), (\alpha, \beta)(\tilde{y}_1, \tilde{y}_2)\big) &= \sqrt{\tilde{D}_1^2(\alpha\tilde{x}_1, \alpha\tilde{y}_1) + \tilde{D}_2^2(\beta\tilde{x}_2, \beta\tilde{y}_2)} \\ &= \sqrt{\tilde{D}_1^2(\tilde{x}_1, \tilde{y}_1) + \tilde{D}_2^2(\tilde{x}_2, \tilde{y}_2)} \\ &= \tilde{D}\big((\tilde{x}_1, \tilde{x}_2), (\tilde{y}_1, \tilde{y}_2)\big).\end{aligned}$$

Suppose λ_1 and λ_2 are the constants from (4) such that

$$\tilde{D}_{1u}(\tilde{f}_1\tilde{x}_1, \tilde{f}_1\tilde{y}_1) = \lambda_1 \tilde{D}_{1u}(\tilde{x}_1, \tilde{y}_1), \quad \text{and} \quad \tilde{D}_{2u}(\tilde{f}_2\tilde{x}_2, \tilde{f}_2\tilde{y}_2) = \lambda_2 \tilde{D}_{2u}(\tilde{x}_2, \tilde{y}_2).$$

Let $\lambda = \min\{\lambda_1, \lambda_2\} > 1, \lambda' = \max\{\lambda_1, \lambda_2\} > 1$. We have

$$\begin{aligned}\lambda\tilde{D}_u\big((\tilde{x}_1, \tilde{x}_2), (\tilde{y}_1, \tilde{y}_2)\big) &\le \tilde{D}_u\big((\tilde{f}_1 \times \tilde{f}_2)(\tilde{x}_1, \tilde{x}_2), (\tilde{f}_1 \times \tilde{f}_2)(\tilde{y}_1, \tilde{y}_2)\big) \\ &= \sqrt{\tilde{D}_{1u}^2(\tilde{f}_1\tilde{x}_1, \tilde{f}_1\tilde{y}_1) + \tilde{D}_{2u}^2(\tilde{f}_2\tilde{x}_2, \tilde{f}_2\tilde{y}_2)} \\ &= \sqrt{\lambda_1^2\tilde{D}_{1u}^2(\tilde{x}_1, \tilde{y}_1) + \lambda_2^2\tilde{D}_{2u}^2(\tilde{x}_2, \tilde{y}_2)} \\ &\le \lambda'\tilde{D}_u\big((\tilde{x}_1, \tilde{x}_2), (\tilde{y}_1, \tilde{y}_2)\big).\end{aligned}$$

Similarly, $\lambda\tilde{D}_s(\tilde{x}, \tilde{y}) \le \tilde{D}_s(\tilde{F}^{-1}\tilde{x}, \tilde{F}^{-1}\tilde{y}) \le \lambda'\tilde{D}_s(\tilde{x}, \tilde{y})$. ∎

Let $p : \tilde{M} \to M$ denote the covering projection. We fix a lift $\tilde{F} = \tilde{f}_1 \times \tilde{f}_2 : \tilde{M} \to \tilde{M}$ of F. Then there is a unique lift $\tilde{G} : \tilde{M} \to \tilde{M}$, that is equivariantly homotopic to $\tilde{F}$.

Now we show that for periodic points x of F and y of G, both of period n, the fact that the orbit of x is K-globally shadowed by the orbit of y is equivalent to that (F^n, x) and (G^n, y) are Nielsen equivalent.

Lemma A.2 (Lemma 1.7 of [Han85])**.** *If x is a fixed point of F^n and y is a fixed point of G^n, then (F^n, x) is Nielsen equivalent to (G^n, y) if and only if $(F, x) \sim (G, y)$.*

Proof. First suppose (F^n, x) and (G^n, y) are Nielsen equivalent, so there exist lifts $\tilde{x}, \tilde{y} \in \tilde{M}$ and a covering transformation t such that $\tilde{F}\tilde{x} = t\tilde{x}$ and $\tilde{G}\tilde{y} = t\tilde{y}$. Then

$$\begin{aligned}\tilde{D}(\tilde{F}^k\tilde{x}, \tilde{G}^k\tilde{y}) &= \tilde{D}(\tilde{F}^{k-n}\tilde{F}^n\tilde{x}, \tilde{G}^{k-n}\tilde{G}^n\tilde{y}) = \tilde{D}(\tilde{F}^{k-n}t\tilde{x}, \tilde{G}^{k-n}t\tilde{y}) \\ &= \tilde{D}(\tilde{F}^{k-n}t\tilde{F}^{-(k-n)}\tilde{F}^{k-n}\tilde{x}, \tilde{G}^{k-n}t\tilde{G}^{-(k-n)}\tilde{G}^{k-n}\tilde{y}) \\ &= \tilde{D}(t'\tilde{F}^{k-n}\tilde{x}, t'\tilde{G}^{k-n}\tilde{y}) = \tilde{D}(\tilde{F}^{k-n}\tilde{x}, \tilde{G}^{k-n}\tilde{y}),\end{aligned}$$

where $t' = \tilde{F}^{k-n}t\tilde{F}^{-(k-n)} = \tilde{G}^{k-n}t\tilde{G}^{-(k-n)}$ is another covering transformation, equal because $F \simeq G$. Thus $\tilde{D}(\tilde{F}^k\tilde{x}, \tilde{G}^k\tilde{y})$ takes on only finitely many values, namely, for $k = 0, 1, \dots, n-1$, so it is bounded.

Conversely, if $(F, x) \sim (G, y)$, then there exist lifts $\tilde{x}$ of x, and $\tilde{y}$ of y such that $\tilde{D}(\tilde{F}^k\tilde{x}, \tilde{G}^k\tilde{y}) \le K$ for some $K > 0$ and for all $k \in \mathbb{Z}$. Suppose $\tilde{F}^n\tilde{x} = s\tilde{x}$ and $\tilde{G}^n\tilde{y} = t\tilde{y}$. Then $s^{-1}\tilde{F}^n\tilde{x} = \tilde{x}$. Since $F \simeq G$, for any $\gamma \in \pi_1(M)$, $G_\#\gamma = F_\#\gamma$. Thus

$$\tilde{D}(\tilde{x}, (s^{-1}\tilde{G}^n)^k\tilde{y}) = \tilde{D}((s^{-1}\tilde{F}^n)^k\tilde{x}, (s^{-1}\tilde{G}^n)^k\tilde{y}) = \tilde{D}(\tilde{F}^{nk}\tilde{x}, \tilde{G}^{nk}\tilde{y}) \le K, \quad \text{for all } k \in \mathbb{Z}.$$

Any bounded subset of $\tilde{M}$ intersects only finitely many lifts of y, and $(s^{-1}\tilde{G}^n)^k\tilde{y}$ is yet another lift of y. There exists an $N \in \mathbb{N}_{\ge 0}$ such that $(s^{-1}\tilde{G}^n)^N\tilde{y} = \tilde{y}$. On the other hand

$$(s^{-1}\tilde{G}^n)^{N+1}\tilde{y} = (s^{-1}\tilde{G}^n)\tilde{y} = s^{-1}t\tilde{y} = (s^{-1}\tilde{G}^n)^N(s^{-1}t\tilde{y}).$$

So

$$s^{-1}t(s^{-1}\tilde{G}^n)^N\tilde{y} = s^{-1}t\tilde{y} = (s^{-1}\tilde{G}^n)^N s^{-1}t\tilde{y}.$$

This implies that $s^{-1}t(s^{-1}\tilde{G}^n)^N = (s^{-1}\tilde{G}^n)^N s^{-1}t$, which implies that G^{nN} fixes the free homotopy class corresponding to $s^{-1}t$. Then by (2'), $s^{-1}t = 1$, so $s = t$. ∎

Remark A.3. Note that only in the very last line of the proof we needed the fact that F is a product of pseudo-Anosov diffeomorphisms and used poperty (2'). Thus we can claim that Nielsen equivalence always implies K-global shadowing.

Lemma A.4 (Lemma 2.1 of [Han85])**.** *(i) If x_1 and x_2 are distinct fixed points of F^n, then (F^n, x_1) and (F^n, x_2) are not Nielsen equivalent; (ii) If x is F-periodic with least period n, then there exists y which is G-periodic with least period n and such that (F^n, x) is Nielsen equivalent to (G^n, y).*

Proof. For (i), suppose (F^n, x_1) and (F^n, x_2) are Nielsen equivalent. There are lifts $\tilde{x}_1$ of x_1 and $\tilde{x}_2$ of x_2 such that $\tilde{F}\tilde{x}_1 = t\tilde{x}_1$ and $\tilde{F}\tilde{x}_2 = t\tilde{x}_2$. Then $t^{-1}\tilde{F}\tilde{x}_1 = \tilde{x}_1$ and $t^{-1}\tilde{F}\tilde{x}_2 = \tilde{x}_2$. $t^{-1}\tilde{F}$ has to be a lift of F that fixes both $\tilde{x}_1$ and $\tilde{x}_2$, but by (4'), there is no lift of any iterate of F that can fix two distinct points.

Now we prove (ii). Let $H : M \times I \to M$ be the homotopy such that $H(x, 0) = F^n(x)$ and $H(x, 1) = G^n(x)$. By (3'), the fixed point index of any fixed point of F^n is never zero, so Theorem 3, p. 94 in [Bro71] states that there exists a fixed point of y that is Nielsen equivalent to x.

It is sufficient to show that y has least period n.

Now fix lifts $\tilde{x}, \tilde{y}$ such that $\tilde{F}^n\tilde{x} = t\tilde{x}$ and $\tilde{G}^n\tilde{y} = t\tilde{y}$, $t \in \pi_1(M)$, so $t^{-1}\tilde{F}^n\tilde{x} = \tilde{x}$ and $t^{-1}\tilde{G}^n\tilde{y} = \tilde{y}$, i.e. $t^{-1}\tilde{G}^n$ is a lift of G^n that fixes $\tilde{y}$. Suppose y has least period $m_1 < n$ and let $m_2 = n/m_1 > 1$. There exist unique lifts $\tilde{F}^n$ and $\tilde{G}^n$ such that $\tilde{F}^n\tilde{x} = \tilde{x}$ and $\tilde{G}^n\tilde{y} = \tilde{y}$. We can find a $t_1 \in \pi_1(M)$ such that $t^{-1}\tilde{G}^n = (t_1\tilde{G}^{m_1})^{m_2}$. Since $t_1\tilde{F}^{m_1}$ is equivariantly homotopic to $t_1\tilde{G}^{m_1}$ and by uniqueness of the equivariant lift, $t^{-1}\tilde{F}^n = (t_1\tilde{F}^{m_1})^{m_2}$. This implies, for any $k \in \mathbb{Z}$,

$$(t^{-1}\tilde{F}^n)(t_1\tilde{F}^{m_1})^k\tilde{x} = (t_1\tilde{F}^{m_1})^{m_2+k}\tilde{x} = (t_1\tilde{F}^{m_1})^k t^{-1}\tilde{F}^n\tilde{x} = (t_1\tilde{F}^{m_1})^k\tilde{x},$$

that is, $t^{-1}\tilde{F}^n$ fixes the entire $t_1\tilde{F}^{m_1}$ orbit of $\tilde{x}$. But we observed by (4') that no lift of an iterate of F can fix two distinct points, so $t_1\tilde{F}^{m_1}$ fixes $\tilde{x}$ and is equal to $t^{-1}\tilde{F}^n$. Therefore $m_1 = n$. ∎

Remark A.5. In the above proof we needed properties (3') and (4').

Lemma A.6 (Lemma 2.2 of [Han85])**.** *There exists a $K = K(G)$ (dependent on G) such that for any $x, y \in M$, if $(F,x) \sim (G,y)$, then $(F,x) \sim^K (G,y)$. In particular, if $x_n \to x, y_n \to y$, and $(F,x_n) \sim (G,y_n)$ then $(F,x) \sim (G,y)$.*

Proof. Let

$$R = \max\left\{\sup_{\tilde{x}\in\tilde{M}} \tilde{D}(\tilde{F}\tilde{x},\tilde{G}\tilde{x}),\ \sup_{\tilde{x}\in\tilde{M}} \tilde{D}(\tilde{F}^{-1}\tilde{x},\tilde{G}^{-1}\tilde{x})\right\}.$$

This maximum is reached and $R < \infty$ because F is homotopic to G, M is compact and the metric is equivariant.

(4') then implies that

$$\tilde{D}_u(\tilde{F}\tilde{x},\tilde{G}\tilde{y}) \geq \tilde{D}_u(\tilde{F}\tilde{x},\tilde{F}\tilde{y}) - \tilde{D}_u(\tilde{F}\tilde{y},\tilde{G}\tilde{y}) \geq \lambda\tilde{D}_u(\tilde{x},\tilde{y}) - R,$$

and similarly $\tilde{D}_s(\tilde{F}^{-1}\tilde{x},\tilde{G}^{-1}\tilde{y}) \geq \lambda\tilde{D}_s(\tilde{x},\tilde{y}) - R$.

Let $K = 2(R+1)/(\lambda-1)$. If $\tilde{D}_u(\tilde{x},\tilde{y}) > K/2$, then

$$\tilde{D}_u(\tilde{F}\tilde{x},\tilde{G}\tilde{y}) - \tilde{D}_u(\tilde{x},\tilde{y}) \geq (\lambda-1)\tilde{D}_u(\tilde{x},\tilde{y}) - R > 1,$$

so $\tilde{D}_u(\tilde{F}\tilde{x},\tilde{G}\tilde{y}) > 1 + \tilde{D}_u(\tilde{x},\tilde{y})$. If $\tilde{D}_s(\tilde{x},\tilde{y}) > K/2$, then $\tilde{D}_s(\tilde{F}^{-1}\tilde{x},\tilde{G}^{-1}\tilde{y}) > 1 + \tilde{D}_s(\tilde{x},\tilde{y})$. This means if any of the distances between the iterates of $\tilde{x},\tilde{y}$ exceeds $K/2$, the orbits cannot globally shadow each other, for any constant K'.

Therefore, if $(F,x) \sim (G,y)$, there must be lifts $\tilde{x},\tilde{y}$ such that $\tilde{D}_u(\tilde{F}^k\tilde{x},\tilde{G}^k\tilde{y}) \leq K/2$ and $\tilde{D}_s(\tilde{F}^k\tilde{x},\tilde{G}^k\tilde{y}) \leq K/2$ for all $k \in \mathbb{Z}$. Then $\tilde{D}(\tilde{F}^k\tilde{x},\tilde{G}^k\tilde{y}) \leq K$ for all $k \in \mathbb{Z}$. We have found a uniform bound, namely K, for the shadowing constant. We can write $(F,x) \sim^K (G,y)$. This K is independent of x, y.

Now we prove the second statement of the lemma. Suppose $x_n \to x, y_n \to y$, and for each n, fix lifts so $\tilde{D}(\tilde{F}^k\tilde{x}_n,\tilde{G}^k\tilde{y}_n) \leq K$ for all k. We claim that there exist convergent subsequences $\{\tilde{x}_{n_j}\}$ and $\{\tilde{y}_{n_j}\}$ that converge to lifts $\tilde{x},\tilde{y}$ so $\tilde{D}(\tilde{F}^k\tilde{x},\tilde{G}^k\tilde{y}) \leq K$ for all k.

Take $U \subseteq M$, a neighborhood of x that is evenly covered by the covering projection p. There is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$. Take $\tilde{U} \subseteq p^{-1}U$ to be the connected component that contains $\tilde{x}_N$, the lift that we picked as above. Then for each $\tilde{x}_n \in \{\tilde{x}_n\}_{n>N}$ there is a $t_n \in \pi_1(M)$ such that $t_n\tilde{x}_n \in \tilde{U}$, so $\{t_n\tilde{x}_n\}_{n>N}$ converges to a lift $\tilde{x}$ of x. We want to show that $\{t_n\tilde{y}_n\}_{n>N}$ contains a convergent subsequence. For any $n > N$,

$$\tilde{D}(t_{N+1}\tilde{y}_{N+1},t_n\tilde{y}_n) \leq \tilde{D}(t_{N+1}\tilde{y}_{N+1},t_{N+1}\tilde{x}_{N+1}) + \tilde{D}(t_{N+1}\tilde{x}_{N+1},t_n\tilde{x}_n) + \tilde{D}(t_n\tilde{x}_n,t_n\tilde{y}_n) \leq 2K + \varepsilon,$$

for a small ε, the diameter of $\tilde{U}$, by the uniform boundedness we showed above and equivariance of the metric. Thus $\{t_n\tilde{y}_n\}_{n>N}$ is a bounded sequence. Therefore, there is a subsequence $\{t_{n_j}\tilde{y}_{n_j}\}$ that converges to some lift $\tilde{y}$ of y.

Now for any $k \in \mathbb{Z}$,

$$\tilde{D}(\tilde{F}^k\tilde{x},\tilde{G}^k\tilde{y}) \leq \tilde{D}(\tilde{F}^k\tilde{x},\tilde{F}^k\tilde{x}_{n_j}) + \tilde{D}(\tilde{F}^k\tilde{x}_{n_j},\tilde{G}^k\tilde{y}_{n_j}) + \tilde{D}(\tilde{G}^k\tilde{y}_{n_j},\tilde{G}^k\tilde{y}) \leq K + 2\varepsilon,$$

for large enough n_j and small ε. Thus $(F,x) \sim (G,y)$. ∎

Remark A.7. In the above proof we needed (4').

Theorem A.8 (Theorem 1 of [Han85])**.** *(i) $(F,x_1) \sim (F,x_2)$ implies that $x_1 = x_2$; (ii) For all $x \in M$, there exists $y \in M$ such that $(F,x) \sim (G,y)$; if x is F-periodic with least period n, then y can be chosen to be G-periodic with least period n.*

Proof. By (4') we have (i).

By (1') periodic points of F are dense in M. For any $x \in M$, there exists a sequence $\{x_n\}$ of periodic points of F, each with least period p_n, such that $x_n \to x$. Then by Lemma A.4 (ii), for each n, there exists y_n that is G-periodic with least period p_n and (F^{p_n}, x_n) is Nielsen equivalent to (G^{p_n}, y_n). By Lemma A.2, $(F, x_n) \sim (G, y_n)$ for all n. Since M is compact, there is a subsequence $\{y_{n_k}\} \subseteq \{y_n\}$ such that $y_{n_k} \to y$ as $k \to \infty$, for some $y \in M$, so $(F, x_{n_k}) \sim (G, y_{n_k})$ and $x_{n_k} \to x$, $y_{n_k} \to y$. By Lemma A.6, we have $(F, x) \sim (G, y)$. If x is F-periodic, the last statement trivially follows from A.4 and A.2. ∎

Remark A.9. Note that in the above proof we used the fact that Nielsen equivalence implies global shadowing to find points that shadow, which does not require property (2'), as said in Remark A.3.

Proof of A.1. Let $Y = \{y \in M : \exists x \in M$ such that $(F, x) \sim (G, y)\}$. For any $x_0 \in M$, by Theorem A.8 (ii), there exsits $y_0 \in M$ such that $(F, x) \sim (G, y)$. Thus we can define a surjective map $\varphi : Y \to M$ by $\varphi(y_0) = x_0$. It is well-defined by Theorem A.8 (i).

Next we show that Y is closed. Take a convergent sequence $\{y_n\} \subseteq Y$ such that $y_n \to y$ for some $y \in M$. For each y_n there is an x_n such that $(F, x_n) \sim (G, y_n)$. Because M is compact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x \in M$. Thus by Lemma A.6, we have $(F, x) \sim (G, y)$ and so $y \in Y$ and Y is closed.

By the above and how we define φ, if $\{y_n\} \subseteq Y$ is such a sequence that $y_n \to y \in Y$, then we have $x_n = \varphi(y_n)$ and $\varphi(y) = x$. Suppose another convergent subsequence $x_{n_k} \to x' \neq x$. Then $(F, x') \sim (G, y)$, which cannot happen because of A.8 (i). Thus $\{x_n\}$ also converges to x, and φ is continuous.

We define $\tilde{\varphi}(\tilde{y}) = \tilde{x}$ for such lifts so we have $\tilde{D}(\tilde{f}^k \tilde{x}, \tilde{g}^k \tilde{y}) \leq K$ for all $k \in \mathbb{Z}$, where K is as in A.6. Then $\tilde{D}(\tilde{\varphi}(\tilde{y}), \tilde{y}) \leq K$ for $\tilde{y} \in \tilde{Y}$, so φ is homotopic to the inclusion.

Finally, take $y \in Y$. Since $(F, x) \sim (G, y)$ implies $(F, F(x)) \sim (G, G(y))$, $F \circ \varphi(y) = F(x) = \varphi(G(y))$. Therefore $F \circ \varphi = \varphi \circ G|_Y$. ∎

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